THE

Mathematics Student

A Quarterly Dedicated to the Service of Students and Teachers of Mathematics in India.

Vol IX.—1941

Edited by

A. NARASINGA RAO, M.A., L.T., D.Sc., F.A.Sc.

Annamalai University, Annamalainagar, South India.

With the co-operation of

N. DURAIRAJAN, M.A., B.E.
B. RAMAMURTI, M.A., D.Sc.
V. GANAPATI IYER, M.A., D.Sc.
S. SIVASANKARANARAYANA PILLAI, D.Sc.
A. A. KRISHNASWAMI AYYANGAR, M.A., L.T.
H. SUBRAMANI IYER, M.A., Ph.D.

. . .

and others.

Printed by REV. FR. A. SUSAI REGIS,
Supt., St. Joseph's Industrial School Press, Trichinopoly

AND

Published by S. MAHADEVAN, M.A.,
Honorary Asst. Secretary, Presidency College, Madras.

1942

CONTENTS

PAGE

Balachandran, K.: Determinants connected with homogeneous products and symmetric functions of the roots of an equation ... ... 137—142
——— : A property of Pascal's triangle ... ... 88— 89
——— : Criteria for an ellipse ... ... 169—170
Banerjee, D. P.: Note on the limit of correlation and regression coefficients in mingled records ... 155—157
Bhalotra, Y.: A criterion for the solubility by radicals of the general quintic ... ... 161—163
Chandrasekharan, K.: The logic of Intuitionistic Mathematics ... ... 143—154
Ganapati Iyer, V.: Topics in the theory of Analytic functions III ... ... 53— 62
Kesava Menon, P.: An expression for Jordan function φ_k (N) in terms of the Euler function φ (N) ... 92
Konnully, A. O.: On the self-polar tetrahedra of three conicoids 133
——— A property of the Tetrahedron ... ... 91— 92
Krishnaswami Ayyengar, A. A.:
——— : The Theorem of Feuerbach ... ... 16— 29
——— : The Triangular Distribution ... ... 85— 87
——— : The Bear clock ... ... 164—165
——— : The greatest angle of a triangle 170
Mandan, S. R.:
——— : F—conic as a rectangular hyperbola ... 170—171
——— : Inconic as a parabola 171
Minakshisundaram, S.: A note on the Theory of Infinite Series ... ... 78— 81
Mitra, A.: A statistical Analysis of the correspondence between the M.A., M.Sc., and B.A., B.Sc., examination results of the Calcutta University in Pure Mathematics ... ... 115—128
Naraniengar, M. T.: Mathematics and life ... ... 6— 7
Pillai, S. S.: On the definition of Oscillation ... ... 165—167
Puri, A.: A method of forming the equation of squared differences ... ... 89— 91
Rajagopal, C. T.:
A proof of Hadamard's Factorization Theorem ... 68— 72
Caratheodory's inequality and allied results ... 73— 77

Rajalakshman, D. V.: On the extreme values of samples taken from a rectangular popul[illegible] ... ... 103—111

Ramanathan, K. G.: On Demlo numb[illegible] ... ... 112—114

Ramesam, Sir V.: A further note on Fe[illegible]ach's Theorem ... 129—130

Rangachariar, V.:

On some properties of rectangular hyperboloids ... 131—133

On the conicoids of a pencil touching a given plane... 158—160

Ranganathan, S. R.:

Prof. M. T. Naraniengar—Reticence incarnate ... 12— 15

Shah, S. M.: A note on the classification of integral functions ... ... 63— 67

Srinivasiengar, C. N.:

M. T. Naraniengar—A Life sketch ... ... 1— 5

Subramani Ayyar, H.: Our first magnitude stars viewed from the Galactic centre ... ... 82— 84

Suryanarayana Iyer: Uniform convergence–a correction ... 167—169

Vijayaraghavan, T.: On Jaina Magic Squares ... 97—102

List of Papers published by M. T. Naraniengar ... 8— 11

Tributes and Reminiscences ... ... 30— 39

The Twenty-eighth session of the Indian Science Congress 41— 42

The Fourth session of the Indian Statistical Conference ... 42

OBITUARY—SIR SHAH MOHAMUD SULAIMAN ... ... 43— 45

QUESTIONS FOR SOLUTIONS 52, 96

SOLUTIONS TO QUESTIONS ... ... 93—95, 136

ANNOUNCEMENTS AND NEWS ... ... 40—41, 135, 174

GLEANINGS 15, 29, 67, 72, 114, 135, 142

BOOKS RECEIVED FOR REVIEW 171

REVIEWS:

C. De La Vallée Poussin—Les nouvelles méthodes de la theorie du potentiel et le Probléme génèralise de Dirichlet ... ... 46— 47

Siddiqi, M. R.—Boundary Problems in Non-linear partial Differential Equations ... ... 47— 50

Gupta, S. R.—Elementary Analytical Dynamics of a particle ... ... 50— 51

Dhawan, M. L.—Contributions to the fundamentals of Arithmetic 51

Archibald, R. C.—Outline of the History of Mathematics 134

Dewan, G. S. and Agashe, D. S.—Differential Equations 135

Ferrar, W. L.—Algebra—a text book of Determinants, Matrices, and Algebraic Forms 171

Godbole, M. S. and Shintre, R. P.—Elementary Plane Trigonometry 172

Raghavachari, T. K.—An introduction to Analytical Geometry and Calculus 173

Author Index of Questions and Solutions

(The Numbers refer to the pages)

Akhaury, V. P.:
Solution 95.

Balachandran, K.:
Questions 94, 96
Solutions 93, 94, 95^2, 175.

Banerjee, G. B.:
Solutions 95^2.

Hansraj Gupta:
Solution 93.

Kerawala, S. M.:
Question 176.

Kesava Menon, P.:
Questions 94, 96,, 136, 175, 176
Solutions 93, 94.

Krishnamurthy, C. R.:
Solution 94.

Krishnaswami Ayyangar, A. A.:
Question 93

Madhusudan Prasad:
Solution 93.

Mahalingam, V. A.:
Solution 95.

Mohd. Khwaja Mohiuddin:
Solution 95.

Narayanan, S.:
Question 176.

Naraniengar, M. T.:
Questions 51^5, 95, 136.

Natarajan, B.:
Solution 95.

Neville, E. H.:
Solutions 136^2.

Puri-A.:
Question 96.

Rangachariar:
Solution 93.

Srinivasan, M. S.:
Question 93
Solution 93

Sundararajan, M. A.:
Solution 95.

Venkatanarasimhan, S.:
Solution 94.

Venkatraman, C. S.:
Solutions 93, 95.

Venkatraman, D.:
Solution 95.

Index of Questions Solved in Volume XI

Question Number	Page	Question Number	Page
478	136	1781	175
1744	93	1791	136
1780	93	1794	94

17232 (Educational Times) p. 95.

ERRATA

Page	line	for	read
64 and 65	The lemma on p. 64 lines 21-24 and p. 65, the footnote marked ‡ on p. 64 and the footnote on p. 65 are not necessary, and should be replaced by the following :— "$f(z)$ cannot belong to maximum type [2, 60]. If $f(z)$ did belong to minimum type, then since $\rho_1 = \rho$; $p = \rho - 1$ it would follow that $q \leqslant \rho - 1$ [2, 90]. This contradicts our hypothesis $q = \rho$. Hence $f(z)$ belongs to mean type."		
71	3 and 4	max (ρ_2, q)	max (ρ_1, q)
,,	15	$\|z_{n_{\nu+\mathrm{I}}}\|$	$\|z_{n_\nu+1}\|$
138	4	side of (5)	side of (4)
,,	7 B†	equations (2)	equations (5)
,,	1 B	H_2	H_1
139	2	with (8)	with (7)
,,	18	Now (11)	Now (10)
,,	19	(7′)	(7)
,,	5 B	After the determinant	insert "......(12′)"
140	4 B	D(n, r)	D (p, r)
141	11	(13) and (13′)	(6) and (6′)
,,	18	(13) and (13′)	(12) and (12′)
144	2 B	Heying	Heyting
145	11	diffrren.	different
151	5 B	makes	make
170	10	V.	K.
172	20	Arya Bhushan Press, Poona	P. T. I. Book Depot, Bangalore
176	10 and 11	$a_n b_1 b_2 \cdots b_n$	$a_n b_1 b_2 \cdots b_{n-1}$

† Read "7th line from the bottom of the page."

Dedicated

To the Memory of

MANDYAM TONDANUR

Naraniengar

FOUNDATION EDITOR, JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY

In Memoriam

शान्तो 'नारायणाचार्यः'
मितवाक् गणिते गुरुः ।
'मध्यविद्यालया'ध्यक्षो
गैर्वाणी वल्लभस्सुधीः ॥

एषा समर्प्यते तस्मै
लसन्ती सूक्तिमञ्जरी ।
तत्प्रीत्यै 'गणिताध्येत्रा'†
साञ्जलिस्सरसा मया ॥

Saṁskṛt (V. A. R.)

† By " The Mathematics Student "

ಶಾಸ್ತ್ರಸಾರವನೆನಿತೊ ಶಿಷ್ಯರಿಗಂದು ಗಣಿತದೊಳರೆದು ಹುಯಿದು,
ಸ್ವಂತ ಸಂಶೋಧನೆಗಳಿಂದಲು „ಮೇಣು " ಹಿಂದೂಗಣಿತಸಂಘದ
ಪತ್ರಿಕೆ"ಯು ಪಲಕಾಲ ಸಂಪಾದಿಸಿದ ಸತ್ಕೈಂಕರ್ಯದಿಂದಲು

ನಮ್ಮ ಹಿಂದೂಗಣಿತಶಾಸ್ತ್ರ ಜ್ಞಾನವನು ಬೆಳೆಸಿ,
ಸಾತ್ತ್ವಿಕತೆಯಿಂ ದೈವಭಕ್ತಿಯಿನಿಹಪರಗಳನು ಸಾಧಿಸಿದರೈ
ಆರ್ಯ ಮಂಡ್ಯಂ ತೊಂಡನೂರಿನ ನಾರಣ್ಣೈಯಂಗಾರ್.

Kannada (G. P. R.)

கண்ணெனத் தகுவ தெண்ணுய ரெண்ணின்
நுண்பொருள் பலவும் நுழைபுலங் கொண்ட
உயர்பே ரறிஞர் உதவிகொண் டினிதா
யயர்வில் உளத்தோ டாண்டுபல வாகக்
கவிஞர் இந்திய கணிதபத் திரிகையைப்
புவியினர் போற்றிப் புகழுமா தந்து
பொறையுடல் விடுத்துப் புகழுடல் எடுத்தோன்
சீரணி கலைப்பெருஞ் செல்வன் சுகுணன்
காரணப் பெயர்பெறு பனவன்
ஆருயிர் சாந்தி யடையவேண் டுவமே.

Tamil. (E. S. V.)

The *Journal*

Empty, she lay, beckoning from afar.
"The world moves on; be ye inane?"
She plaintive cried; "O Bhatta, Gupta, Bhāskar,
Be these indeed your kith and kin?"
Then V. R. thundered out "We Are!"
While M. T. N. cooed "empty ēn?" †

English. (A. N. R.)

---

† ēn (என்) in Tamil, (M. T. N's native language) means "why."

M. T. NARANIENGAR
AGE 40.

AGE 69.

THE MATHEMATICS STUDENT

Volume IX] MARCH 1941 [Number 1

M. T. NARANIENGAR: A LIFE SKETCH.

BY

DR. C. N. SRINIVASIENGAR, *Engineering College, Bangalore.*

Mandyam Tondanur Naraniengar was born on the 8th October 1870 at Melkote (Tirunarayanapuram), a Sree-Vaishnavite religious centre in the Mysore State. His father, Sri Prativadi-bhayankaram Singlacharya was a well-known scholar and worker in the cause of the Vishistadwaita system of philosophy. But the family was not sufficiently well-off; moreover, owing to the affliction caused by the death of a number of children, one of them a boy at the age of 24, Mr. Singlachar lost all interest in the ordinary affairs of the family. He took these incidents so much to heart that for years afterwards he confined himself more or less to his house. Under such circumstances was M. T. Naraniengar born, and the family then consisted of two brothers* and a sister.

The Aksharabhyasam (initiation into the alphabet) of the two brothers was performed by a maternal uncle Sri Srirangacharya, the father being reluctant to attend to any celebration, owing to the mishaps in his family. The brothers studied Kannada and Sanskrit in the Sanskrit Pathasala (school) at Melkote till the end of 1879. The brothers had their Upanayanam function celebrated at this time in far-off Srirangam. Soon after, in 1880 a philanthropic gentleman of Madras, Sri Vakil Krishnaswamy Iyengar, who was a great Sanskrit scholar, took Naraniengar with him and put him to an English School at Triplicane (Madras). Naraniengar's elder brother joined him at Madras shortly after. But their stay at Madras was rather short, since the brothers began to feel that they should not make their unfortunate parents feel their separation, and decided to return to Melkote. But friends and relatives prevailed upon the parents, and the education of the boys was soon resumed. This time, Naraniengar was sent to Hassan (Mysore State) under the care of a near relative, while his elder brother went to Shimoga.

* The elder brother of Naraniengar was M. T. Narasimhiengar who retired as Assistant Professor of Sanskrit in the Central College, Bangalore, and is the author of a number of religious works in Sanskrit.

The brothers progressed well thereafter. Naraniengar passed the Matriculation Examination in 1883 from the High School at Hassan. He had meanwhile been married (betrothed) in 1882 in his 12th year, and his wife's uncle Sri M. A. Appalacharya at Mysore enabled him to complete the course for his F. A. Examination, which he passed in 1885. Naraniengar always stood first in the class and was the recipient of scholarships. The earnings of the two brothers from scholarships and private tuitions gave them a decent sum of about Rs. 45/- which enabled the parents to settle down at Bangalore and continue their sons' education in the Central College, Bangalore for the B. A. Degree. Naraniengar studied Mathematics under Professors R. H. Piggot and T. R. Venkataswami Naidu, and took his B. A. Degree in Mathematics in 1888, standing first in the First Class in the Madras University Examination for the B. A. Degree held in January 1888. As a student, Naraniengar made a good impression on his teachers, particularly Prof. Piggot, who often acknowledged that Naraniengar's solutions to problems were shorter than his own. In turn, Prof. Piggot's lucid and attractive exposition of the subject left its mark on the would-be Professor of Mathematics; and Piggot's beautiful handwriting was mainly instrumental in Naraniengar's developing that exquisite handwriting which is a matter of admiration to all who have seen it.

After Naraniengar passed his B. A. Degree examination, Prof. Piggot tried to send him to Cambridge for the Mathematical Tripos. But family circumstances and religious sentiments were in the way, and Naraniengar proposed to study for Engineering either at Madras or at Poona. But just then an additional mathematics assistant was wanted by the Principal of the Central College and having been selected for the post, Naraniengar, reported himself for duty as Mathematics Assistant in the Central College, Bangalore on the 16th January 1889 in his nineteenth year on a pay of Rs. 70 per month. He continued in this capacity till February 1894, when Prof. T. R. Venkataswami Naidu was appointed Professor of Mathematics in the vacancy caused by the death of Prof. Piggot, and then Naraniengar was appointed Assistant Professor. Naraniengar had meanwhile passed the M. A. Degree examination, and he took the Degree in 1894. Shortly after, Prof. Venkataswami Naidu was transferred as Assistant to the Inspector-General of Education, and Naraniengar became Professor of Mathematics on the 25th February 1896, which post he held till his retirement in October 1925. As a teacher, his teaching was efficient and inspiring. Four hours of continuous class-work would not strain him or ruffle his temper.

He was very sympathetic towards the students, and he gave financial assistance to many towards their studies. By his colleagues he was held in great esteem for his scholarship, character and sociability.

During the early years of his service, Naraniengar's mathematical ability brought him in contact with Sir K. Seshadri Iyer, the then Dewan of Mysore. Sir K. Seshadri Iyer was a keen student of mathematics, and had an idea of establishing an astronomical observatory in Mysore, to be placed in the charge of Naraniengar. A large number of books on astronomy were made available to Naraniengar for study, and the two persons often met and discussed problems on the subject, and spent long hours in the study of the Heavens. Sri K. Seshadri Iyer used to take Naraniengar on long walks, discussing diverse problems such as State Provident Fund, Insurance schemes etc. The astronomical observatory did not materialize, but astronomy always remained for Naraniengar a favourite subject for teaching. He also kept up his walking, and a minimum of five miles of walk per day became an almost invariable part of Naraniengar's daily time-table.

Two keen and able students of mathematics were brought together when the late V. Ramaswami Aiyar was for a brief period in 1894 a colleague of Naraniengar in the central college. The two became deeply immersed in geometry, and would often, while walking along, stop in the road "to draw and discuss a figure, careless of mankind passing by." Later when in 1907, Sri V. Ramaswami Aiyar started the Indian Mathematical Society, both Prof. Naraniengar and Prof. Venkataswami Naidu (who had by then become Professor of Mathematics at the Maharaja's College, Mysore) became foundation members and found themselves on its managing committee. In the progress reports of the Indian Mathematical Club for October 1908 were printed the first two original mathematical papers published in India—the first "On the Cardioid" by Prof. R. P. Paranjpye, the second, "The Nine-Point Circle" by Prof. M. T. Naraniengar. Prior to this, several Indian mathematicians including Naraniengar as well as V. Ramaswami Aiyar had been contributing to mathematical periodicals in England, and a few papers of Naraniengar had been published in the *Proceedings of the Edinburgh Mathematical Society*, and the *Mathematical Gazette*. The *Journal of the Indian Mathematical Society* was formally started in 1909 and Naraniengar became its editor, working also as one of the Secretaries. He continued as Editor of the journal for a period of 18 years till June 1927; during the later years he had a Joint Editor to assist him.

No praise will be too much for the services which Naraniengar has rendered to the Indian Mathematical Society. In assessing his services to the Society, one has to bear in mind the pioneering nature of the work, the poor facilities that existed in those days for printing mathematical matter, and the inexperience of the mathematical researchers whose work though of high order lacked the necessary qualities such as brevity, clarity and proper finish. Rewriting of papers and of questions and solutions, and preparation of diagrams drawn to scale for making blocks involved considerable labour and skill. Time had to be devoted to these as well as to proof-reading and routine administrative work of the Society in addition to the long hours which the professor had to spend in the class-rooms at the College. Nor was this all. Naraniengar himself contributed very frequently papers, short notes, questions and solutions. His mathematical work ranges over a variety of subjects, and one sees in them an uncanny sense of intuition and mathematical skill.

Naraniengar was elected President of the Indian Mathematical Society for the session 1930-32. During the Silver Jubilee celebrations of the Society at Bombay in 1932, the Indian Mathematical Society honoured him with an Address. The text of the Address and Prof. Naraniengar's reply are published in the Jubilee Volume (Vol 20—First Series) of the *Journal of the Indian Mathematical Society.*

In private life, Naraniengar was a simple unassuming man, with remarkable character and sense of discipline. His official dress was in an exceptionally beautiful Indian style. He kept up the high traditions of his ancestors in his piety, devotion to God, study of Philosophy and Sanskrit, and was actively associated with many religious and social institutions. He has rendered yeoman service in the improvement and the working of the Ubhaya Vedanta Pravartana Sabha, the Sri Yathiraja Mutt of Melkote, the Mandyam Sree Vaishnava Sabha, the Mandyam Vidyarthi Nilayam, and many other institutions. He took great pains in order to start a Sanskrit Academy in Mysore, but he did not live to see it inaugurated. Whether it was in the administrations of these bodies, whether it was in the work of the Board of Studies in Mathematics, the University Council or the Senate, whether it was in his work as editor, writer or teacher, thoroughness and sincerity of purpose always characterized his work. He commanded the respect of all.

His domestic life was marred by several unhappy events. Very early, he lost a promising young son-in-law, and between the

years 1920 and 1926 he lost three grown-up sons including his eldest. Marvellous was indeed the way in which he was able to control his grief, and resume normal work. His study of philosophy was not in vain!

In stature he was tall and lean. It was an interesting contrast to the on-lookers when at the Bangalore Conference of the Indian Mathematical Society in 1926, the two most prominent members of the Society, Sri V. Ramaswamy Aiyar and Prof. M. T. Naraniengar were weighed at one of the excursions that had been arranged, and the balance recorded 246 lbs. and 98 lbs. respectively! During his early days of service, Naraniengar played tennis for some time, but walking was his only physical exercise afterwards. His health in his last year of life was rather poor, but he kept up all his activities. He had a stroke of paralysis on the morning of 4th October 1940 from which did not recover at all, and passed away peacefully at 4 P.M. on the 9th October 1940. He leaves behind him his wife, two sons and a daughter. To the Biblical age of three score and ten, he added one day in deference to the Hindu custom of adding one to all figures, good and noble.

MAY GOD BLESS HIS SOUL!

[This Life Sketch, especially as regards details of Prof. Naraniengar's early days, is based upon an elaborate life-history of the family written by his elder brother, the late M. T. Narasimhiengar. I owe my thanks to Sri M. T. Ramaswami Iyengar, M.Sc. for making this life-history of the family available to me. I am also obliged to him for his painstaking work in the preparation of the accompanying list of publications. I am also indebted to Mr. V. Gopalaswami Aiyangar, M.A., (Retired Registrar of the Mysore University) who was a colleague of Prof. Naraniengar as Assistant Professor of Mathematics for several years, for his giving me several valuable points of information as well as for carefully scrutinizing and improving this life sketch—C. N. S.]

MATHEMATICS AND LIFE*

Is it really possible to treat of questions in Life mathematically by graphs? The answer is: If questions of general Psychology can be so investigated, there is no reason why our query should not be answered in the affirmative.

We speak of life in this world as belonging to one plane and that in another world as belonging to a different plane. Let us briefly examine whether there is any justification for this figure of speech, borrowed from the geometer's phraseology.

The life of an individual consists of a chain of events depending upon environments, which in their ultimate nature are a condition of time. And time is generally regarded as a one-dimensional continuous quantity. Further, it must be admitted that in the chain of events (forming the life of an individual) any event is inseparably and inevitably linked to the preceding event. Thus every event is a corollary to the preceding event. Hence, taking time as the independent variable, there ought to be a functional relation between the event and time. This functional dependence may take the form $y=f(x)$, where x is the independent variable viz. time, and y the dependent event. The nature of the function f varies, of course, with the individual. Strictly speaking, the constants which enter into the function f are themselves of the nature of arbitrary constants and may therefore be regarded as so many independent variables, thus complicating the problem. For our present purpose, we shall take equation $y=f(x)$, as representing the life of an individual approximately.

Consistently with the above convention, the following equivalents may be proposed to express the analogy between Life and Mathematics :

(1)	Incidents in Life	=	Points on the Life curve
(2)	Course of events	=	Gradient of the curve
(3)	Environments	=	Points in the neighbourhood of the curve
(4)	Influence of Environments	=	Relation of neighbouring points to the curve

* Part of an Address delivered by Prof. M. T. Naraniengar before the Bangalore Mathematical Association on Wednesday the 19th March 1919. A few of the statements may perhaps be controversial, but the whole forms interesting reading. [C. N. S.]

(5)	Contacts between two individuals	=	Contacts between the corresponding curves
(6)	Intercourse between two individuals	=	Intersection of the corresponding curves
(7)	Turning Point of Life	=	Turning Point of the curve
(8)	A group of Individuals	=	A group of curves
(9)	A Nation	=	A family of curves
(10)	The national ideal	=	The characteristic Property of the family
(11)	Courses of conduct determining an event	=	Line co-ordinates
(12)	Uniform steady Life	=	The straight line
(13)	Ideals of Life	=	Asymptotes to the curve
(14)	The Law of Birth	=	Periodicity of curves
(15)	Existence Beyond	=	Complex variables
(16)	Singularities in Life	=	Singular Points on a curve
	etc.		etc.

We shall conclude with a quotation from a very interesting book on *Mathematical Psychology* by Mrs. M. E. Boole (pp. 68-9).

"Most people are content to be swayed to some extent by personal afflatus. But as tangential direction leads at last to drifting away from the centre of force, safety for everyone depends on keeping not too far from the centre: everyone should treat the direction in which he feels impelled to go as one co-ordinate, and the path which leads towards those who seem to him in error as the other co-ordinate; and he should accordingly correct his own aberrations by acts of conversion towards the centre.

Therefore, the sacrifice appropriate to any given situation of difficulty or doubt is that which converts the force which is taking the individual along his habitual path into *energy of union with opponents.* The centre of progress is not on the line which the opponents are following, but on the line which runs from us to our opponents."

LIST OF PUBLISHED PAPERS

Progress Reports of The Indian Mathematical Club

1. The Nine-Points circle ... ... No. 7, pp. 10-14.
2. Notes on Spherical coordinate Geometry ... No. 8, pp. 4-9

The Journal of the Indian Mathematical Society

	Volume			Pages.
3.	,,	I.	Eulers' line	46-7
4.			A Note in Analytical Geometry ...	90-2
5.			Note on Questions 35 and 49 of Mr. V. Ramaswami Aiyar	
6.			Notes on Spherical coordinate Geometry ...	127-30
7.			On Question 144 of Mr. V. Ramaswami Aiyar	236-37
8.			Poles and Polars in Point and Line Coordinates	240
9.	,,	II.	Triangular Poles and Polars	8-13
10.			The Gergonne Point	60
11.			Root-Pedals and Root-curves	99-104
12.			Centre of Pressure	147-9
13.			Sphero-conics	168-78
14.	,,	III.	Flexion in Polar Coordinates	26
15.			The Foci of an in-conic	66-71
16.			The locus of points at which the opposite sides of a quadrilateral subtend equal or supplementary angles	132-40
17.			Bicircular Quartics	152
18.			A Trigonometrical Note	237-8
19.	,,	IV.	The sixfold system of similar circumscribed and inscribed triangles of a given triangle	21-3
20.			Some properties of curves and surfaces ...	84-90
21.			The Harmonic Centre	96-8
22.			Approximations in curve tracing ...	162-70
23.	,,	V.	The Harmonic centre	14
24.			A set of Simultaneous Equations ...	61
25.			The three-cusped hypocycloid and its spherical analogue	82-9
26.			Properties of polynomials	176-8
27.			On factors	220

28.	,, VI.	The spherical mean centre	3–11
29.		The normal section of an enveloping cylinder	61–3
30.		Stereographic projection	137–41
31.		Note on the diameter of a quadrilateral ...	180
32.	,, VII.	Cyclic Equations	7–13
33.		The Generalised Cassinian	59–62
34.		Cyclotomic Equations	62–3
35.		A Special Class of Equations	82–8
36.		The axes of a conicoid and an allied cone	134
37.		A Geometrical Proof of a Trigonometric Identity	142
38.		The Folium of Descartes	202–8
39.		Note on planetary aberration	213
40.	,, VIII.	The conical envelope of a conicoid ...	13
41.		Note on Partial Fractions	51–2
42.		Newton's Theorems	53–4
43.		The Period of the elliptic functions sn u, cn u, dn u	98–9
44.		Holditch's Theorem	100–1
45.		The roots of a derivative of a rational function	135–6
46.		Geometric representation of a definite Integral	137
47.		The Legendre Expansion of $F\{1-2r\cos\theta+r^2)^n$	163–6
48.		Note on Question 742	180
49.		Centroid of a uniform circular arc ...	216
50.	,, IX.	Note on Maxima and Minima	46–8
51.	,, X.	Note on Question 735	269
52.		Complex roots of equations	294-309
53.		A Diophantine Problem	348
54.	,, ,,	An interesting cubic connected with an ellipse	390
55.	,, ,,	Degenerate bicircular quartics ...	391
56.	,, ,,	A geometrical problem	476–8
57.	,, XI.	Fractional differentiation	88–95
58.	,, ,,	On Questions 942 and 943 (Editorial) ...	34
59.	,, XII.	Notes on the equation $a\sec\theta+b\operatorname{cosec}\theta=1$	60
60.	,, ,,	*Leaves from a Lecturer's Diary ...	139–40
61.	,, ,,	* Do	180–1
62.	,, XIII.	* Do	13
63.	,, XV.	A Note in Analytical Geometry ...	49–51

* These anonymous notes are presumably those of Mr. M. T. Naraniengar, and are characteristic of his method of work.—*C. N. S.*

The Mathematics Student

64. „ IV. Presidential Address at the Seventh Conference of the Society held at Trivandrum in April 1931. (*Report of the Seventh Conference*)

65. „ „ V. Ramaswami Aiyar, the Founder of the Society—A Life Sketch. 111–6

Journal of the Indian Mathematical Society

Serial Numbers of Questions Proposed.

Volume	
Prog. Rep. 8.	8, 14, 39, 48.
I.	79, 86, 90, 112, 141, 148, 150, 165, 173.
II.	184, 203, 204, 212, 224, 225.
III.	271, 272, 298, 306, 307, 334.
IV.	351, 416.
V.	478, 491.
VI.	545.
VII.	614.
IX.	875.
X.	959.

Serial Numbers of Questions Solved.

2 (P.R. 7), 6 (P.R. 8), 16 (P.R. 7), 14 (P.R. 8), 27 (I), 148 (II), 203 (IV).

The Mathematical Gazette

	Volume			Page
1.	„	IV.	Two geometrical proofs of Feuerbach's theorem (one of them by Mr. V. Ramaswami Aiyar)	311
2.	„	V.	The Cycloid	337
3.	„	„	The Cubic $xy^2 + 4a^2 = 0$	340
4.	„	„	On some problems and results	389

Proceedings of the Edinburgh Mathematical Society

Vol. XXVI	(1907–08)	The intrinsic properties of curves.	87–92
Vol. XXVIII	(1909–10)	The locus of points at which two sides of a given triangle subtend equal or supplementary angles ...	73–80

Mathematics from the Educational Times (New Series)

Vol. XV (1909) A Proof of Feuerbach's theorem ... 31

Questions Proposed

II. 17127.
XIV. 16300, 16382, 16411.
XV. 16339, 16469.
XVI. 16617.
XVII. 16500, 16598, 16648
XVIII. 16629, 16726, 16762, 16767, 16803
XIX. 16821, 16835, 16887, 16966
XX. 16339, 16726, 16835, 16869, 16940, 16966, 16974, 17011, 17039.
XXI. 17073.
XXII. 17268.
XXIII. 17137, 17215.
XXIV. 16966, 17182, 17232
XXV. 17251.

Questions Solved

XII. 16065, 16087, 16103, 16105, 16138, 16147, 16157, 16165.
XIII. 16156, 16158, 16200, 16208, 16229, 16248, 16265, 16270, 16297.
XIV. 16245, 16246, 16285, 16288, 16308, 16310, 16336 16342, 16392.
XV. 8670, 11614, 11759, 16336, 16357, 16381, 16382, 16390. 16429, 19437, 16444, 16446, 16472, 16477, 16478, 16490.
XVI. 10585, 11289, 16471, 16544, 16547, 16552, 16557, 16571, 16576, 16579.
XVII. 11604, 16337, 16489, 16525, 16570, 16575, 16627, 16633, 16657, 16694, 16708.
XVIII. 16466, 16797, 16816.
XIX. 12085, 16215, 16726, 16815, 16833, 16835, 16856, 16877.
XX. 11592, 16921, 16983.
XXIII. 11403.
XXIV. 16106, 17160.
XXV. 17161.

Prof. M. T. NARANIENGAR, RETICENCE INCARNATE

BY

RAO SAHIB S. R. RANGANATHAN, M.A., L.T., F.L.A.,
Librarian, Madras University Library
and
Secretary, Madras Library Association

The beginnings of the mathematics honours course of the Madras Christian College were as meagre as they could be. It was begun in July 1913 with one student in the fourth year and two in the third. As the sole occupant of the fourth year class I had all the complexes of an only child of a family—in particular, intolerance and jealousy. One factor that induced jealousy centred round Prof. M. T. Naraniengar, whose name had been alluring me, for long, from its prominent position on the cover pages of the successive issues of the *Journal of the Indian Mathematical Society*. One morning the plucky one of the two juniors of mine began to spin out a long yarn about M. T. N.—he discussed elliptic functions with him; he perused the file of papers M. T. N. had for the meeting of the Board of Studies;...he had, as a child, played on the lap of M. T. N.! This was enough to conjure in my mind an ardent desire to see and hear the Professor. But my jealousy of the unusual good fortune of my plucky junior became simultaneously so intense that I would not accept his offer to take me to the professor. And so I had to bide my time. Since then I had heard M. T. N's voice but once, although I had met him thrice.

First Sight

The earliest occasion when I met Prof. M. T. Naraniengar was on the 27th December 1916 when the mathematicians of India met for the first time in conference in the English lecture hall of the Presidency College, Madras. That conference produced a lasting impression on most of the lads who were my contemporaries at college. Many names rendered familiar in the pages of the *Journal of the Indian Mathematical Society* turned out to be realities for the first time. But on the opening day the spendour of the Governor's presence and the peculiar headwares of Prof. Sanjana, Prof. Kapadia, Prof. Swaminarayan and others from other parts of India and the bubbling movement and omnipresence of Prof. V. Ramaswami Iyer

had completely overshadowed M. T. N. It was on the 28th morning that I could get a close view of him. All the delegates had assembled at about 7-30 a.m. in what is now called the Fyson Park. Mr. K. S. Patrachariar played the monitor's role self-appointed and unofficial, of course. With his characteristic out-spokenness he was calling out the names from the list of members and I took advantage of it to know the who's who. When M. T. N's name came, there was no response; but I had to break my four years' resolve and humble myself to ask my plucky junior who M. T. N. was. He pointed out to a tall, thin figure standing all alone. The unusually thick long long-coat which made him look taller and thinner than what he probably was, the Mysore turban which functioned as his crown and the small but tightly closed lips made a permanent impression in my mind.

Second Sight

It was about 8-30 a.m., January 9, 1919. The Bombay Mail steamed into the Adoni station at which I was waiting for it. I had described elsewhere* the voice that pulled me into the compartment into which were crowded most of the reputed mathematicians of Madras who were on their way to the second Mathematical Conference to be held at Bombay. There were in the group Prof. Ramaswami Iyer, the founder of the Indian Mathematical Society, who virtually overshadowed everybody else. There was Prof. P. V. Seshu Iyer, showing his practical wisdom at every turn, who was only second to Ramaswami Iyer in his loquacity. Another gentleman who occasionally added his voice to the conversation was Mr. S. Narayana Iyer. But the tall slim figure with a long long-coat and a Mysore turban had still its thin lips absolutely pursed up. I wondered if they would have ever parted since I saw them last at Madras! I saw Prof. M. T. Naraniengar on all the three days of the Conference at Bombay, at meetings and social gatherings, in the Elephanta caves and on the boat which took us to and from the caves; but still I could not get even one occasion to hear M. T. N's voice. Even the humorous songs of Patrachariar which elicited peels of laughter from everybody else on the steam launch felt powerless in breaking open the lips of M. T. N. His bright face adorned with the *namam* and his black long-coat relieved by the white *angavastram* added a dignity to his tall slim form, which was, I believe, enhanced considerably by his persistent reticence throughout

* *The Mathematics student.* V. 4. 1936. P. 145.

the three days of the Conference—at least whenever I happened to be near him anxiously waiting for an opportunity to hear his voice.

Words Though Not Voice

It was in 1922 that my thought was once again switched on to this reticent professor. This time it was not his sight nor his voice that did so. It was his written words. How I wished they were spoken words! About that time Prof. Seshu Iyer and others had plotted to break a tradition in the university circle. Till then all the university lecturers were drawn only from the senior members of the staff of the colleges. It would appear that Prof. Ramaswami Iyer had insisted that the younger members should be given a chance. The choice fell on me. I was then reading—not reading but plodding through—the Theory of Groups. At the suggestion of Prof. Seshu Iyer, I drafted a syllabus for a course of university lectures on that subject. Some days later, Prof. Seshu Iyer read to me from a letter of Prof. M. T. Naraniengar which rightly said: "There is no doubt that juniors should be given a chance and that the Theory of Groups is a most fitting subject. The syllabus is all right; but the most fitting junior to whom the course should be entrusted is Vaidyanathaswami." The letter had also given an estimate of my mathematical capacity as measured by that of Dr. Vaidyanathaswami, which latter events had so well confirmed. I mention this not merely because I have at present no mathematical reputation to lose but because it proves beyond doubt the unsparing judgement and fearless certitude of Prof. M. T. Naraniengar which are usually lacking even in academic worlds.

I must however confess that at the moment Prof. M. T. N's decree, though it did not prevent my being the first junior to become a university lecturer, induced some uncontrollable bitterness in my mind; from that day whenever there was delay in the publication of my little notes or solutions to problems in the *Journal*, I used to attribute it foolishly to the turning down of my contributions by the editorial prejudice of M. T. N. But to my great discomfiture the whole lot of my contributions would appear in the next issue of the *Journal*. Recurrence of such experiences began to wash out my bitterness little by little.

Voice at Long Last

I had said that I had heard his voice once. I should certainly refer to that occasion. It was in June 1931. I was in Bangalore on

my way to Nandi hills. I was staying with a cousin of my wife. Among those who came to visit me, there was a mathematical friend.

He asked "Should you not look up M. T. N." I said "I do not know him—rather he does not know me." But my friend said "You are thoroughly mistaken. I mentioned to him last evening that you were here. He told me many details about you and, although he did not say that in so many words, I could very well see how eager he was to meet you." I said that I no longer belonged to mathematics. But my friend replied "The professor was talking keenly about your new work and your work and life in England and so on."

All this pleading on the part of my friend made me try one more chance to hear the voice of M. T. N. and so I went to Malleswaram. I was delighted to see the tall slim figure, sans turban, sans the long long-coat. The bright face adorned by the lovely *namam* looked brighter as it was in its native glory unhampered by the conventional saratorial encumbrances. I bowed to him. The small red lips parted into a genial smile and he put several questions in melodious Mysore Tamil.

At long last the desire of 1913 was fulfilled in 1931.

GLEANINGS

He (Euclid) merely postulates, in the abstract, that one can draw a line through 2 points, or a circle about a point, without devoting a single word to *how* one does it. Here Euclid is doubtless under the influence of the notion which prevailed in certain ancient schools of philosophy, that practical application of a science was something inferior, artisan-like. Unfortunately this view persists in many places today, and there are still always university teachers who cannot be too scornful of any concern with applications, as being ignoble. The arrogance of such views should be vigorously combated. We should value equally highly every admirable performance, whether in the theoretical or in the practical field, and we should allow each individual to concern himself with those things to which he feels most strongly inclined. In this way, any person will show himself the more versatile the more talent he posesses. The most eminent mathematicians, as Archimedes, Newton, Gauss, have always uniformly included both theory and applications.

FELIX KLEIN,

Elementary Mathematics from an advanced standpoint: *Geometry*

(*Elementar Mathematik von hoheren Standpunkt aus*: Bd. II.)

THE THEOREM OF FEUERBACH

BY

A. A. KRISHNASWAMI AYYANGAR, *Maharajah's College, Mysore.*

1. As a tribute to the memory of M. T. Naraniengar, let me offer this multi-coloured flower of perennial beauty, which was first produced by Feuerbach and round which other flowers were later gathered in the course of a century of active mathematical thought. K. W. Feuerbach (1800-1834) published in 1822 a pamphlet on the geometry of the triangle, in which he found the distances between every pair of remarkable points and the radius of the circle through every triad of notable non-collinear points. In this process, he hit upon what is now the most famous theorem in all elementary geometry, viz., that the circle through the feet of the altitudes passes through the middle points of the sides and touches the inscribed and escribed circles. Indeed these six points are the essential points of the circle, for in a modified form they generalise as Hart's circle on the sphere.(13) Some decades elapsed before the three '*points Euleriens*', the middle points of the distances between the orthocentre and the vertices were discovered on the circle and its designation was immortalised as the nine-points circle. It may be remarked here that if the orthocentre be replaced by any general point, the corresponding set of nine points will lie on a conic and therefore, if any *five* of the nine-points lie on a circle, the remaining *four* points also will lie on the same circle, thus showing how the majority of *five* out of the nine points determines the fate of the residual minority of *four* points.

2. The spirit of Feuerbach was fully appreciated by four great geometers of the Indian Mathematical Society, M. T. Naraniengar, A. C. L. Wilkinson, V. Ramaswami Aiyar and M. Bhimasena Rao and I dare say that the contact of these four geometers with Feuerbach's theorem has added more glory to it than the contact of the four tri-tangent circles to the nine-points circle. The pioneer Indian touch to the theorem was given by M. T. Naraniengar in an article of five pages in '*The Progress Report of the Indian Mathematical Club*' dated 15th October 1908. It forms the humble source of the great stream of geometrical thought running in the pages of the full blown journal. The two initials affixed to Naraniengar's

name seem to signify that he was the Master of the Trilinears, a tool handled by him with such skill and power in unravelling geometrical mysteries of all sorts, be it the nine-points circle, or the sphero-conic (*J. I. M. S.* 1910, 168-178), or the tricusped hypocycloid (*J. I. M. S.* 1913, 82-89) or the Folium of Descartes (*J. I. M. S.* 1915, 202-208). He was a concrete mathematician intent on fixing up definitely the co-ordinates of all his points and the equations of all his lines and circles, so as to give as complete a census as possible of the mathematics of the region he explored. Thus, he gave the trilinear co-ordinates of the four Feuerbach points, wrote down the equations of their joins in pairs, showed that they met the sides on the bisectors of the angles of the triangle[1], calculated the co-ordinates of the poles of the sides of the Feuerbach quadrangle with respect to the nine-points circle, identified the diagonal points of this quadrangle as the intersections of the corresponding sides of the pedal triangle and the medial triangle† and discussed all the collineations, concurrences, parallelisms, perpendicularities and apolarities that he met in his way.

The nine-points circle family is hopelessly endogamous with its various associated triangles, points, lines and circles involving Simson-line relations and orthopolar relations. The properties being extremely dense, lend themselves easily to multiple re-discoveries. It is no wonder, therefore, that some of Naraniengar's results had been anticipated by earlier workers, while others were re-discovered several years later. As a rapid preliminary survey, Naraniengar's work is valuable and has been a source of inspiration for younger mathematicians.

3. Coolidge[18] has remarked that the number of proofs in existence, of Feuerbach's theorem is transfinite, the last adept being probably Sawayama with his nine proofs published in 1911 (*L'Enseignement Mathematique,* Vol 13). R. Goormaghtigh[22] has also given a list of proofs and references to bibliographical notes. I will now briefly indicate the various lines of attack adopted by mathematicians in this fertile field, classifying their work under three heads:

(i) elementary proofs, (ii) algebraic proofs, and (iii) generalisations.

The following general conventions will be adopted for convenience:

ABC is the fundamental triangle of sides a, b, c; A'B'C' the medial triangle, DEF the orthocentric pedal triangle, S the circumcentre,

† This property appears to have been re-discovered by Deaux (*Mathesis,* 1922, p. 282) as Goormaghtigh's note ([23]) would indicate.

R the circum-radius, O the orthocentre I the in-centre, r the in-radius, P the in-Feuerbach point, and N the centre of the nine-points circle of ABC; A_1, B_1, C_1 the middle points of OA, OB, OC; U, V, W the middle points of IA, IB, IC; K, L, M the points of contact of the in-circle with the sides BC, CA, AB; the same letters with suffixes a, b, c denote corresponding points with respect to the three escribed circles. (XY, α) denotes a circle through X, Y, and Z such that $X\hat{Z}Y = \alpha$; to avoid ambiguity as to the side of XY on which the segment containing 'α' lies, $X\hat{Z}Y$ is defined as the angle through which ZX turns in the anti-clock wise direction so as to coincide with ZY. With this convention, (XY, α) and (YX, $-\alpha$) will mean one and the same circle.

4. *Elementary proofs of Feuerbach's Theorem.*

(1) A really elementary proof ought not to make use of ratios or trigonometrical formulæ. The best elementary proof, therefore, is a pure 'angle-chaser' (in the picturesque phrase of Prof. E. H. Neville), provided to us by W. F. Beard (*M. G.*† 1934; 322, 323). This proof may be modified and condensed thus:

Let the nine-points circles of BIC, CIA meet again in P having already a common point W.

Since $A'\hat{P}W = I\hat{C}B$, $W\hat{P}B' = A\hat{C}I$,

$$A'\hat{P}B' = A'\hat{P}W + W\hat{P}B' = A\hat{C}B;$$

hence P lies on the nine-points circle of ABC. (i)

$$\text{Further, } K\hat{P}L = K\hat{P}A' + A'\hat{P}B' + B'\hat{P}L$$

$$= C\hat{B}I - I\hat{C}B + A\hat{C}B - A\hat{C}I + I\hat{A}C$$

$$= C\hat{B}I + I\hat{A}C = K\hat{M}L. \quad \text{(ii)}$$

Hence P lies on the pedal circle of I, i.e. the in-circle.

Again, $D\hat{P}A' = C\hat{B}A - A\hat{C}B = 2\,(C\hat{B}I - I\hat{C}B) = 2K\hat{P}A'$,

i.e. DK, KA′ subtend equal angles at P.

† Abbreviation for '*Mathematical Gazette*'.

Hence, a straight line PT such that $T\hat{P}D = P\hat{A'}D$ will be a tangent to both the in-circle and the nine-points circle at P.

Similar proofs can be devised for ex-circles.

There are two important general theorems at the background of the above proof:

(*a*) Bobillier's theorem that the pedal circle of a given point on a conic with respect to any inscribed triangle passes through a fixed point, which becomes the centre of the conic, when it is a rectangular hyperbola.(8)

(*b*) V. Ramaswami Aiyar's theorem that the pedal circle of any point Q w.r.t. ABC cuts the nine-points circle at an angle equal to the complement of the sum of the angles BAQ, CBQ, ACQ.

Mannheim's theorem(22) and its equivalent, Naraniengar's question 165 in *J. I. M. S.* 1919 may be re-stated more elegantly in the form that the circles on UK, VL, WM as diameters all pass through the in-Feuerbach point. This result also follows immediately from the proof given above.

(2) Other elementary proofs by A. Ranganatha Rao (*J. I. M. S.* Notes and Questions, 1931, 10), H. N. Haskell (*M. G.* 1937, 154; and 1940, 214), E. P. Lewis (*M. G.* 1940, 108), and W. J. Dobbs (*M. G.* 1939, 291) as well as the familiar proofs in Trigonometry text-books utilise rectangle properties and similar triangles in addition to the two special properties:

(i) DA′ subtends at any point on the nine-points circle an angle equal to $(C\hat{B}A - A\hat{C}B)$ or its supplement; and

(ii) $A'K^2 = A'D . A'D'$ where D′ is the intersection of AI and BC.

In fact, one can manage with these two properties alone as shown by K. J. Sanjana (*J. I. M. S.* Notes and Questions, 1929, 16). Sir V. Ramesam tells us that there are two elementary proofs due to V. Ramaswami Aiyar and M. T. Naraniengar, which are practically identical (*Mathematics Student* †, 1936, 123).

(3) There are two standard proofs by the method of inversion, one due to Casey(6) who inverts the nine-points circle into the side

† Henceforth condensed to M. S. I have no access to these proofs.

BC, and the other due to Fontene[16], who inverts the same circle into the fourth common tangent of the in-circle and the ex-circle opposite to A (or two ex-circles). Both of them choose the radius of inversion so that the in-circle and an ex-circle (or two ex-circles) invert into themselves. Since the inverses obviously touch, the originals also do so. Before Fontene, J. Young had given a proof[9] almost parallel to Fontene's inversion method. H. N. Haskell purposely evades inversion and camoufluages Young's method, while W. J. Dobbs and T. Krishna Rao (*J. I. M. S.* 1917, 228) do so with Casey's method by following a circuitous course of elementary reasoning, ultimately based on the fact that the line joining corresponding points on the in-circle and the nine-points circle meets either circle in a point which also lies on the other circle.

(4) There are two other proofs due to Casey, one based on the property of common tangents to four circles which touch a fifth[3] and the other based on the theory of three similar figures[5].

(5) Mc Cay's proof[4] shows that the Simson-lines of the extremities of the diameter SI of the circle ABC meet on the nine-points circle at a point whose* distances from A′, B′, C′ are proportional to the tangents from the same points to the in-circle and therefore, the point is a limiting point of the two circles. The latter part of the proof contains a property of co-axal circles and is a favourite item in the arguments for proving Feuerbach's theorem. The different forms of the argument run as follows.

(*a*) The radical axis of the nine-points circle and the in-circle touches either circle and therefore the two circles touch. This principle is also used in analytical methods. R. P. Paranjpye's proof (*J. I. M. S.* 1911, 193) identifies the tangent at a point P on the nine-points circle whose distances from A′, B′, C′ are proportional to $|b-c|$, $|c-a|$, $|a-b|$ with the radical axis of the in-circle and the nine-points circle. It exhibits two important properties of the Feuerbach point:

(i) the distances of the Feuerbach tangent from the middle points of the sides are equal to $(b-c)^2/8NI$, etc. and (ii) the distances of the Feuerbach-point from the same points are $|b-c|\sqrt{R/8NI}$, etc ; results which have been several times rediscovered in the literature of Feuerbach's theorem (*M. G.* 1936, 145-148).

(*b*) A limiting point of the nine-points circle and the in-circle is a point P, which satisfies the condition A′P : B′P : C′P = A′K : B′L : C′M

and lies on the nine-points circle, because $a.\ A'P \pm b.\ B'P \pm c.\ C'P = 0$, the signs being suitably determined according to the position of P †

(*c*) Again $a.\ A'K \pm b.\ B'L \pm c.\ C'M = 0$ with a suitable combination of signs is a sufficient condition for inferring the contact of the nine-points circle and the in-circle. One may recognise this as a particular case of Casey's condition for four circles (here three points A′ B′ C′, and a circle) to touch a fifth circle (See § 5 (2) below).

5. *Analytical Proofs*:

Analytical methods provide a delightful mathematical adventure in the Feuerbach field. Several mathematicians including M. T. Naraniengar, A. C. L. Wilkinson, (*J. I. M. S.*, 1917, 55-73), and N. M. Gibbins (*M. G.* 1938, 97-104) have played good tunes by 'turning the handle of the barrel organ of algebraic geometry.'

(1) The classical method is to show, as we have already indicated, that the radical axis of the nine-points circle and the in-circle touches the in-circle and similarly for ex-circles; and M. T. Naraniengar departs slightly from this procedure by showing that the point which divides NI in the ratio $R : 2r - R$ lies on the in-circle, with similar proofs for ex-circles.

(2) An alternative method is to use Purser's condition(10) $at_1 + bt_2 + ct_3 = 0$ (§4 (5c) above) for the circle in areal coordinates

$$(x + y + z)(xt_1^2 + yt_2^2 + zt_3^2) - \sum a^2yz = 0$$

to touch the circumcircle of the triangle of reference. The areal equation of the in-circle (referred to the medial triangle) is

$$(x + y + z) \sum x\,(b - c)^2 - \sum a^2yz = 0$$

and the condition for tangency is obviously satisfied.

(3) There are two articles(14) by Cayley on the nine-points circle, where the circle is treated as a particular case of the bi-punctual conic in areals

$$\sum pp'\ x^2 - \sum (qr' + rq')\ yz = 0 \qquad \text{(i)}$$

which cuts the sides BC, CA, AB in (P_1, P_2), (Q_1, Q_2) (R_1, R_2) such that the six lines such as AP_1, AP_2, meet in two points †† $(1/p, 1/q, 1/r,)$ $(1/p', 1/q, 1/r')$.

† Compare *M. G.* 1930, 213, where Hilton and Neville adopt a similar method for proving an extension of this theorem.

†† By oversight, Cayley gives the reciprocals of these co-ordinates.

If the conic be referred to the medial triangle, the equation (i) changes into

$$\sum x. \sum x \; (q-r)(q'-r') - \sum (q+r)(q'+r') \; yz = 0, \qquad \text{(ii)}$$

which will reduce to a circle, when

$$a^2 : b^2 : c^2 = (q+r)(q'+r') : (r+p)(r'+p') : (p+q)(p'+q') \qquad \text{(iii)}$$

The geometrical interpretation for (iii) is that the centres of the in-conics $P_1 Q_1 R_1$, $P_2 Q_2 R_2$ are isogonal conjugates in ABC.

Further, the circle will touch the nine-point circle if

$$\sum a \sqrt{(q-r)(q'-r')} = 0$$

i.e. $$\sum \sqrt{(q^2-r^2)(q'^2-r'^2)} = 0$$

i.e. $$\sum p^2 \; (q'^2 - r'^2) = 0, \text{ on rationalisation.} \qquad \text{(iv)}$$

M. Bhimasena Rao has proved that the condition for the circle (ii) to touch the nine-points circle is that the line joining the centre of the in-conic $P_1Q_1R_1$ and its isogonal conjugate in ABC should pass through the symmedian point of ABC. He calls the circle (ii) the contact-circle with respect to the centre of the in-conic $P_1Q_1R_1$. (*J. I. M. S.* Notes and Questions 1927, 49-51.)

(4) Professor H. F. Baker[25] gives an algebraic proof of Feuerbach's theorem on the projective plane, where ω, ω' are the absolute points of the plane.

$\omega \, I \, \omega'$ being taken as the triangle of reference, the equation of the in-circle may be written $xz - y^2 = 0$; (i)

if the points of contact K, L, M be denoted by the parameters α, β, γ, the equation of BC is

$$x - 2y\alpha + z\alpha^2 = 0, \qquad \text{(ii)}$$

and the equation of the circle A′ B′ C′ is obtained as

$$(xz - y^2)(rp_1^{-2} - \mu) + y \; (x - 2y\mu + z\mu^2) = 0 \qquad \text{(iii)}$$

where $p_1 = \sum \alpha$, $q_1 = \sum \alpha\beta$, $r_1 = \alpha\beta\gamma$, $\mu = q_1/p_1$

It is now evident from (iii) and (i) that the circles touch each other at P $(\mu^2, \mu, 1)$.

The values of the parameters α, β, γ are the roots of the cubic $\theta^3 - p_1 \theta^2 + q_1 \theta - r = 0$, while the parameters of ω, ω', and P are the

roots of $p_1\ \theta^2 - q_1\ \theta = 0$, which may be regarded as a cubic with roots ∞, 0, μ.

Hence, we may state that the nine-points circle touches the in-circle at a point on it which is the † apolar complement of the circular points at infinity with reference to the points of contact of the in-circle with the sides.

(5) Prof. L. J. Rogers (*M. G.* 1930, 111) proves a generalisation of Feuerbach's theorem by using complex co-ordinates. This method may as well serve to prove Feuerbach's theorem itself.

The circumcentre S of the triangle ABC being taken as the origin, let the complex co-ordinates of A, B, C and a point equidistant from the sides be Rα, Rβ, Rγ, Rz, so that $|\alpha| = |\beta| = |\gamma| = 1$; and if $p_1 = \sum\alpha$, $q_1 = \sum\alpha\beta$, $r_1 = \alpha\beta\gamma$, then the orthocentre is given by Rp_1 and the nine-point centre by $\frac{1}{2}$Rp_1.

The equation of BC may be written

$$z + \bar{z}\ \beta\gamma = \mathrm{R}\ (\beta + \gamma)$$

and the length 'ρ' say, of the perpendicular on BC from the point 'Rz' is

$$\frac{\mathrm{R}}{2} \cdot \frac{z + \bar{z}\beta\gamma - \beta - \gamma}{\sqrt{\beta\gamma}}.$$

Hence, after some simplification, we get

$$\alpha(\bar{z}^2 r_1 + 2z - p_1) + \beta\gamma\ (1 - 2p_1\bar{z} + q_1\bar{z}^2 + 2z\bar{z} - 4\rho^2/\mathrm{R}^2)$$
$$+ z^2 - p_1 r_1 \bar{z}^2 - 2zp_1 + 2\bar{z}r_1 + p_1^2 - q_1 = 0$$

with two similar equations got by cyclic exchange in α, β, γ.

This implies that we may equate to zero the coefficients of α, $\beta\gamma$ and the symmetrical terms.

Hence, $4\rho^2/\mathrm{R}^2 = 1 - 2p_1\bar{z} + q_1\bar{z}^2 + 2z\bar{z}$ ... (1)

$p_1 = \bar{z}^2 r_1 + 2z$ ... (2)

$q_1 = z^2 + 2\bar{z}r_1$ ... (3)

† Two cubics $a\theta^3 + 3b\theta^2 + 3c\theta + d = 0$, $a'\theta^3 + 3b'\theta^2 + 3c'\theta + d' = 0$ are said to be apolar when $ad' - 3bc' + 3b'c - a'd = 0$.

Substituting from (2) and (3) in (1), we get

$$4\rho^2 = R^2 (1 - 2z\bar{z} + z^2\bar{z}^2) \qquad \dots (4)$$

$$\text{i.e.} \quad |z| = \sqrt{1 \pm \frac{2\rho}{R}}$$

The distance of N from the in-centre or an ex-centre is given by

$$R|\tfrac{1}{2}p_1 - z| = \tfrac{1}{2}R|z|^2 \text{ from } (2)$$

$$= \frac{R}{2} \pm \rho \text{ from } (4),$$

the proper sign being selected from geometrical considerations.

(6) An analytical proof by the method of invariants is suggested by J. H. Grace (*J. I. M. S.*, 1917, 95). If Γ be a rectangular hyperbola through the vertices A, B, C, whose centre lies on the in-circle Γ', then the invariant relations give $\theta'^2 = 4\Delta'\theta$, and $\theta = 0$, and therefore $\theta' = 0$, which shows that the centre of Γ' lies on Γ. Hence Γ passes through A, B, C, I and is thus uniquely fixed.

Indirectly, the nine-points circle of ABC which is the locus of the centres of rectangular hyperbolæ circumscribing ABC has only one common point with the in-circle, and therefore the two circles touch—a proof obtained as it were by circumstantial evidence.

(7) Hamilton's proof[2] is worth noting also. It is based on the following general theorem :

If two conics Γ, Γ' touch four common tangents at (a, b, c, d), $(a'\ b',\ c'\ d')$ while a third conic Ω through a, b, c touches Γ' at d', then the second chord of intersection of Ω and Γ' is the line through the intersections (aa', bc), (bb', ca), and (cc', ab).

Replace a, b, c by A′, B′, C′; Γ by the Steiner ellipse which touches the sides of ABC at the middle points, and Γ' by the in-circle; then Ω becomes the nine-points circle, for a common chord of Ω and Γ' is the line at infinity. This shows that the Feuerbach tangents touch the maximum inscribed ellipse also.

6. *Generalisations.*

Feuerbach's theorem lends itself admirably to expansion and so every inch of the theorem has been generalised. It is outside the scope of this paper to give a full account of all the generalisations that have been achieved. One can barely hint at a few remarkable ones.

(1) At the very threshold of the cyclic property I may suggest the following:

Let PQR be a triangle of species (α, β, γ) inscribed in ABC, another triangle. By a well known property, the circles AQR, BRP, CPQ meet in a point which may be denoted by $O_{\alpha\beta\gamma}$ whose isogonal conjugate in ABC is $O'_{\alpha\beta\gamma}$.

THEOREM:—If PQR be any inscribed triangle of ABC, then the $O'_{\alpha\beta\gamma}$, points of the outlying triangles AQR, BRP, CPQ lie on the circumcircle of PQR.

The proof of the theorem is immediate and I proceed to the corollaries:

(i) If PQR is similar to ABC, the orthocentres of the outlying triangles lie on the circle PQR.

(ii) If PQR is similar to the orthocentric pedal triangle of ABC, then the circumcentres of the outlying triangles lie on the circle PQR.

These two results include the property of the nine-points circle.

(iii) If PQR is similar to the contact triangle of the in-circle of ABC, the in-centres of the outlying triangles lie on the circle PQR.

I have noticed in *J. I. M. S.*, 1919, 240, that if PQR is any triangle inscribed in another triangle ABC, the orthocentres of the outlying triangles and P, Q, R, lie on a conic and the area of the triangle formed by the orthocentres is equal to that of PQR. This raises the query:

Do any other set of corresponding points of the outlying triangles and P, Q, R, lie on a conic?

(2) The budget of generalisations may be arranged in three groups, each centering round certain basic ideas.

GROUP I: *Auxiliary circles, director circles, and joint director circles of confocals:*

(*a*) If the axis of a conic inscribed in a triangle passes through the circumcentre, its auxiliary circle touches the nine-points circle (M' Kay[7])

(*b*) If the axis of an in-conic of a triangle passes through Q, the corresponding auxiliary circle and the pedal circle of Q are co-axal with the director circle of the in-conic touching the axis at Q (S. Narayanan, *J. I. M. S.*, 1918, p. 419.)

(c) If a conic is inscribed in a triangle and one of its confocals passes through the circumcentre, then the mutual orthoptic circle of the two conics touches the nine-points circle. (V. Ramaswami Aiyar, *Edinburgh Mathematical Society Proceedings* 1896.)

(d) If a pair of isogonal conjugates w.r.t. a triangle are also conjugate w.r.t. the circumcircle of the triangle, their pedal circle cuts orthogonally the nine-points circle of the triangle. (Shusaku Ogino, *Tohoku Mathematical Journal*, Vol. 43, p. 147.)

(e) If $\sum$ be a conic inscribed in a triangle and $\sum'$ the confocal to $\sum$ in-polar to the circumcircle of the triangle, the mutual orthoptic circle of $\sum$ and $\sum'$ cuts orthogonally the nine-points circle of the triangle. (K. Rangaswamy, *M. S.*, 1938, 155.)

All the above theorems are readily deduced from two key-theorems.

(i) If P and Q are two isogonal conjugates in a triangle and S the circumcentre, $(R^2 - SP^2)(R^2 - SQ^2) = 4R^2q^2$, where q is the length of the semi-minor axis of the in-conic with P, Q as foci. (Genese's theorem, p. 76, Gallatly's *Modern Geometry*.)

(ii) If M is the middle point of PQ in the above theorem and N the nine-point centre, then SP. SQ = 2R. MN. (V. Ramaswami Aiyar's Theorem, p. 79, Gallatly, l. c.)

GROUP II: *Pedal circles and orthopolar conics:*

(a) The pedal circle of every point on a line w.r.t. a triangle ABC cuts a fixed circle orthogonally. The centre of this circle, called the orthopolar circle, is the orthopole of the line w.r.t. ABC, and the square of the radius is equal to twice the rectangle contained by the distances from the line, of its orthopole and the circumcentre (Lemoyne [15]). This leads immediately to Bobillier's theorem for the rectangular hyperbola already noticed and M'Kay's extension of Feuerbach's theorem, when the orthopolar circle dwindles to a point.

(b) Lemoyne's theorem was generalised by Bricard in 1920 thus: If S_1, S_2, S_3 are tangential equations to three conics, the auxiliary circles of the conics $K_1S_1 + K_2S_2 + K_3S_3 = 0$ having a focus on a given straight line are orthogonal to a fixed circle.[19]

(c) Ramaswami Aiyar's generalisation is that if X be any conic inscribed in the triangle ABC, and Y a confocal touching LL', then the mutual orthoptic circle of X and Y cuts the orthopolar circle of LL' orthogonally. (*J. I. M. S.*, 1912, 214, 215).

(*d*) In 1910, Neuberg proved that the locus of the orthopoles of straight lines through any given point w.r.t. a triangle is an ellipse.[21] The auxiliary circles of the conic touch the nine-point circle and the major and* minor axes of the orthopolar conic are the diameters perpendicular to the Simson lines of the extremities of the diameter through the given point, of the circumcircle of the fundamental triangle.[24]

Recently, K. Rangaswami has given a synoptic view of most of the above generalisations with the help of his key theorem, viz:

(*e*) If Γ be a range of conics and Γ′ any other range formed by conics confocal with those of Γ, the mutual orthoptic circles of corresponding confocals in Γ, Γ′ belong to a co-axal system. (*M. S.*, 1938, 93).

GROUP III: *Hart's system of circles and circular curves having the nine-points circle as their envelope.*

(*a*) A triangle formed by three circular arcs is called a circular triangle, and the circle which touches the inscribed circles of a circular triangle and its associated triangles is called the Hart's circle of the triangle. It has several properties analogous to those of the nine-points circle. For example, the Hart circle for a triangle ABC cuts the sides at angles equal to the differences of the angles of the triangle.

If we denote the circles BC, CA, AB by X, Y, Z and the Hart's circle of ABC by H, we can prove that the circles X, Y, Z, H form a system such that each is the Hart circle of one of the triangles formed by the other three circles.[11] J. H. Grace suggests a possible analogous property for tri-circular sextics, viz. that the tri-tangent circles of a tricircular sextic can be arranged in sets of four all tangent to the same circle. (*J. I. M. S.*, 1917, 98)

Recently, Dr. A. Narasinga Rao has studied the principle of duality in circle, sphere and line geometries (*Math. Zeitschrift* 1938) which when applied to the configuration of two Hart Tetrads of circles leads to the generalisation (26) that there exist on a plane dual sets of four circles such that each circle of one set has with each circle of the other set a chord of given length. When the chords shrink to zero length, we get the contact case.

Again, in his paper 'Studies in Circle Geometry' (*M.S.* 1940, 68-72), the same author discusses the possibility of having Hart

pentads of spheres in 3-space and spheres in n-space which cut $n+2$ given spheres at the same angle.

In this connection, reference may be made to a noteworthy article by Dr. R. Vaidyanathaswami, 'On four mutually orthogonal circles', (*Proceedings of the Cambridge Phil. Soc.*, Vol 24, part 4).

(*b*) A generalisation of Feuerbach's theorem to oriented circles is given by Bricard[17] thus:

If $l_1l'_1$, $l_2l'_2$, $l_3l'_3$ be three pairs of oriented lines in involution, the four oriented circles properly tangent respectively to the triads $l_1l_2l_3$, $l_1l'_2l'_3$, $l'_1l_2l'_3$, $l'_1l'_2l_3$ are properly tangent to a fifth oriented circle.

Feuerbach's theorem follows if we specialise the pairs as the corresponding sides of a triangle and its medial triangle.

(*c*) Lastly, Morley's theorem (*J. I. M. S.* 1920, 54-58) deserves mention: Every curve of class three that touches the six joins of four orthocentric points and passes through the circular points at infinity touches the nine-points circle.

In the course of proving this theorem M. Bhimasena Rao. (*J. I. M. S.* 1924, 208) has proved two allied theorems, one of which (i. e. the second one) is due to Ramaswami Aiyar.

(*d*) If PQ the diameter of a rectangular hyperbola passes through the nine-point centre of an inscribed triangls ABC, every bicircular quartic which has P as double focus and passes through P and the feet of the perpendiculars from P on the six joins of A, B, C, H (the orthocentre) in pairs, touches the nine-points circle at the centre of the rectangular hyperbola.*

(*e*) Given a triangle ABC, it a point P and the feet of the perpendiculars from it on the sides and altitudes of ABC all lie on a Cartesian Oval, then the negative pedal of the oval w.r.t. P touches the nine-points circle. (*J. I. M. S.* 1918, 371)

References

(1)† JOSEPH WOLSTENHOLME: *Mathematical Problems*, 1878 ; † 222 (Question 1319).

(2) GEORGE SALMON : *Conic Sections*, 313.

* I have slightly modified Bhimasena Rao's enunciation to make it more intelligible to the general reader.

† The references indicate the pages in the works cited. The bracketed numbers give the citations in serial order.

(3) WILLIAM J. MC'LELLAND: *Geometry of the Circle*, 1891, 11 and 13; (4) 183; (5) 217.

(6) JOHN CASEY: *A Sequel to Euclid*, 1886, 105, 106; (7) *Analytical Geometry*, 1893, 329; (8) 541.

(9) R. LACHLAN; *Modern Pure Geometry*, London, 1927, 74, (10) 206; (11) 254; (12) 283.

(13) CAYLEY *Collected Mathematical Papers*, Cambridge, 1897, 551; (14) 517-521.

(15) WILLIAM GALLATLY; *The Modern Geometry of the Triangle*, Second Edition, 46-54.

(16) J. L. COOLIDGE: *Treatise on the Circle and the Sphere*, Oxford, 1916, 41; (17) 375, (18) *Bulletin of the American Mathematical Society;* 1929, 19.

(19) R. GOORMAGHTIGH: The Orthopole: *The Tohoku Mathematical Journal*, 1926, 81; (20) 82; (21) 87; (22) 101, 102; (23) 106.

(24) A. A. KRISHNASWAMI AYYANGAR: A simple exposition of the allied properties of the Simson-line, the Steiner envelopes and the orthopole: *Educational Review*, Madras 1924.

(25) H. F. BAKER: *Principles of Geometry*, Vol. II, 1922, 113, 114.

(26) A. NARASINGA RAO: On a principle of duality in circle, Sphere and Line Geometries: *Mathematische Zeitschrift*, 1938, 194.

GLEANINGS

Numbers and Letters; these are the two eyes of a full and prosperous existence.

TIRUVALLUVAR: *Tirukkural.*

TRIBUTES AND REMINISCENCES

From Dr. R. Vaidyanathaswami, Madras

President of the Society (*1941*)

By the demise of Professor M. T. Naraniengar, the Indian Mathematical Society is losing one more of that enthusiastic group of Mathematicians who started the society and developed its activities during its earlier years, and who were always ready to lend their active support and co-operation to the Society. As the foundation-editor of the *Journal* for a period of nearly twenty years, Prof. Naraniengar discharged almost single-handed the difficult and responsible task of conducting the *Journal.* His own contributions were not inconsiderable and exhibit a wide range and scholarship. Of the two, Prof. Naraniengar and V. Ramaswami Aiyar, who were the original pillars of the society, the former was able to supply, in virtue of his academic position, that erudition and all-round information which the latter, in spite of his originality and energy, could not well command in his service as an executive officer. One can hardly talk of these two separately as it was their combination and co-operation which was the great asset of the society in the early days when mathematical research in India was at its infancy. I have no doubt that their names will live in the annals of our Society and be treasured by all lovers of mathematics.

From Dr. R. P. Paranjpye, Minister of Education Bombay Presidency (Retd.)

President of the Society (*1936-1940*)

The late Prof. M. T. Naraniengar was associated with the Indian Mathematical Society from the very beginning. When the late Mr. V. Ramaswami Aiyar first conceived the idea of the Society—he first called it the Indian Mathematical Club—he had the enthusiastic cooperation of Prof. Naraniengar. The journal of the Society was first started in a very modest form and from that time Prof. Naraniengar was its editor. He did that work very efficiently. He had a very wide and deep knowledge of the subject, and this is very necessary for an editor as he has to judge of the merits and originality of the papers submitted for insertion in the journal. It involved a vast degree of labour to bring out the journal regularly, and Naraniengar gave it very cheerfully. The Indian Mathematical Society is under a deep debt of gratitude to him for his devoted work.

I was not very intimate with him personally though I met him three or four times at Bangalore when I was staying there for six months in 1925 and later at one or two of the Conferences of the Society. He appeared to be a man of very simple habits and orthodox in his views. The large number of his pupils whom I met in Bangalore and elsewhere spoke with affection and regard of their teacher.

From the files of the journal a better account can be given of his mathematical work than I can do from memory. He himself contributed several papers but he criticised and improved many more as the editor of the journal. This latter work is now beyond record and can only be recalled by the authors of the papers themselves.

His death is a very great loss to the Indian Mathematical Society.

From Prof. R. N. Apte, Kolhapur

President of the Society (1912-1915).

The late Prof. M. T. Naraniengar with unselfish devotion conducted the Journal of the Indian Mathematical Society very ably. He was a brilliant mathematician and made valuable original contributions to the journal. In the early days it was very difficult to edit such a journal, but our Professor carried on the work with remarkable zeal and energy. In the matter of problems sent for publication he took great care to see that they were not to be found in already published sources. His memory and vast knowledge of such problems was wonderful. It was through him that young students of mathematics gradually realised the need for sending original problems and thus he gave a great stimulus to mathematical study. In his last days his penetrating and keen intellect worked in the domain of philosophy from a mathematical point of view and his published thought in this subject may very well be compared with those of Jeans and Eddington.

From Sir C. V. Raman, Indian Institute of Science, Bangalore

Having been connected with the Indian Mathematical Society for over thirty years, I consider it a privilege to associate myself with the movement to do honour to the memory of the late Professor M. T. Naraniengar by bringing out a special number of *The Mathematics Student.* Since coming to Bangalore seven years ago, I have had frequent occasions to come into personal contact with him and appreciate his high scholarly and personal qualities. Professor Naraniengar was deeply interested in scholarly movements of various kinds. He was the President of an organisation which examined and recognised Sanskrit learning of the traditional type. The Indian Mathematical Society has reason to be grateful to him for the devoted services he rendered to the Society during his long association with it.

From Sir V. Ramesan, Retired Judge, High Court, Madras.

It is with great regret that I heard the death of Prof. M. T. Naraniengar from a letter of Prof. Narasinga Rao. Of his abilities as a Mathematician I first heard from our common friend the late Mr. V. Ramasami Iyer. In my opinion both must be regarded as joint founders of the Indian Mathematical Society. When I first met him in 1910 I was very much struck by his extreme simplicity.

Immediately after the Society was founded progress reports were being issued. In the report No. 7 dated 15th Oct. 1908, we have an article on the nine point circle by Prof. Naraniengar in which he determines the trilinear co-ordinates of the Feuerbach points and thence proves the famous theorem, and then follow a number of interesting propositions nine in number about the Feuerbach points. The whole of this paper was summarized and reproduced in Gallatly's *Modern Geometry of the Triangle* pp. 37-38. From the time the *Journal* was started he was its chief editor and was one also of the Hon. Joint Secretaries. He continued to be editor of the *Journal* up to Vol 16 and thereafter he was a member of the Committee of Collaborators. During these 16 years every volume of the *Journal* contained important contributions by him.

The last time I met him was in Oct. 1938 when he presided in connection with an address by me on the Modern Geometry of the Orthopole in the Central College, Bangalore. He was very much enfeebled by ill health.

His geometrical proof of Feuerbach's theorem appeared in the *Mathematical Gazette*, London, Vol. 4, page 311, along with another by the late Mr. Ramaswami Iyer. The two proofs are connected though not identical.

The progress of Mathematics in Southern India was very much indebted to his devotion to the subject and has sustained a great loss by his death.

From Prof. C. R. Narayan Rao, Bangalore

Editor : " Current Science "

I came in contact with Prof. M. T. Naraniengar in 1909, when I joined the Mysore Educational Service as a member of the Central College Staff. On the day I met him, I encountered simultaneously three other members of the staff belonging to his community, and naturally I concluded that the Central College staff consisted of Brahmins of one particular sect. I thought within myself that these gentlemen had mistaken their profession and had accidentally strayed into a scientific institution, while their proper place was in a Sanskrit Academy. On recovering from the bewilderment engendered by the array of orthodoxy before me, I selected the senior member for paying my respects first, and the response was, " You are Narayan Rao ? " This unsophisticated style of greeting laid the foundation of close association with him, whom to know was a pleasure, and to work with, a privilege.

Joseph Louis Lagrange is reported to have maintained that a mathematician has not thoroughly understood his own work till he has made it so clear that he can go out and explain it effectively to the first man he meets on the street. This ideal may not be in the grasp of most mathematicians, but I have reasons to believe that the late Prof. Naraniengar had very nearly attained it. I should, however, like to say a few words about him as a friend and colleague. Great as he was as a mathematician, he was even greater as a gentleman. I have never known him speak ill of others, even under provocation. He would never compromise with truth, nor palter with principles. He had a sturdy independence of mind, and he would advocate a cause, however unpopular and unacceptable, were he convinced about its righteousness. As a man he possessed the most lovable traits of character. He feared God, and was always full of kindness. He was not effusive, but underneath that somewhat stern exterior there beat the warmest human heart. There was a nature, contrary to what many might believe who did not know him intimately, of the uttermost simplicity. His intellectual interests were divided between Mathematics and Hindu philosophy. India of the early ages spoke to him through her sacred book, as she speaks to all her sons who have ears to hear, and the key to the late Professor Naraniengar's life is to remember that his roots had struck deep into the religion of his ancestors, and it was from an early India that he drew his inspiration and the sources of his strength and the purity of his character. His students will remember him as an eminent Professor and his friends and colleagues, as an excellent gentleman.

From Dewan Bahadur Dr. S. Krishnaswami Aiyangar, Madras

M. T. Naraniengar came of an orthodox Brahman family of Vaishnavas of the Mandayam community in Mysore. Born in the year 1870, he came to the Central College from Mysore for his B. A. course, taking Mathematics for his optional subject. He was among a small number of students who formed the first batch under the late Mr. Pigot of the Central College, who united in himself, a rare combination for those times, specialist knowledge of Mathematics, coupled with qualifications for the Professorship of English in a first-grade College. His first-appointment in the Mysore service was as Vice-Principal of of the Central College, with the full responsibility of an English Professorship and the sole responsibility for Mathematics. Almost from the beginning of his unfortunately short career, he created a very good impression as a professor and proved to be quite as good as the impressions formed of him, though his valuable services were lost to the college in the course of a couple of years or three. From the very beginning of his career, he was taken up with the lad M. T. Naraniengar, a nice looking lean boy, precociously intelligent in mathematics in a class of eight. He paid special attention to the brilliant boy to coach him up to the required level in English rather than in Mathematics, and when Naraniengar's application had been sent up, he took special pains along with the other English Professor, the late Mr. C. M. Leonard, to coach Naraniengar in English. Naraniengar got through the examination with ease, and

stood first for the presidency in Mathematics, which did not come with any surprise to his professors or principal, all their anxieties having been about his getting through in English, possibly an exaggerated fear. Almost immediately Naraniengar was taken into the Education Department of the Central College as Assistant Professor of Mathematics, and his life's career was thus made without any ado, notwithstanding the efforts made by the late Mr. Pigot, his Professor, to persuade him to go to Cambridge. Naraniengar came of an orthodox family, and the times were orthodox as well, and he had to lose the great chance of distinction which he would have attained had he proceeded to Cambridge, though perhaps it would have been distinction of a different kind from that of the late S. Ramanujam. Naraniengar soon attracted the attention of the remarkable Dewan, Sir Seshadri Aiyar, whose partiality for brilliant students was well known, and who took sufficient interest even to prescribe work and getting Naraniengar to study various things connected with his special subject of study. Soon after his entry into service Professor Pigot died, and the late T. R. Venkatasami Naidu became Professor in the central college with Naraniengar as his Assistant. Afterwards, Prof. Venkatasmi Naidu was transferred as Inspector of Schools, and Naraniengar was promoted to the professorship. He continued to do good work and was one of those silent professors, who did their work without making fuss or creating any ill feeling through his long career of service, including the difficult period of the principalship of the late Dr. Cook, who in spite of all that was good in him, was exacting as a taskmaster, and whose satisfaction was difficult to secure except by hard work. It speaks volumes for the efficiency of Naraniengar that he got on with the principal who had claims to knowledge of Mathematics in addition to his own physics. Silent and unobtrusive as Naraniengar was, he was in his own quiet way acceptable to his colleagues generally and popular among the students. In his own unostentatious way he managed to do some quiet work in his own subject apart from the teaching, and made a name for himself both at the University and among his mathematical colleagues. When later the Indian Mathematics Society was brought into existence, he easily secured for himself a place among the front ranks of those who had to bear the brunt of the work of the Society. He was at one time the President of the Society and did hard work to edit the journal as well. As a teacher he was efficient and as a worker in mathematics, he won ungrudging praise from all quarters competent to form an opinion about his work.

As a man Naraniengar was quiet and unostentatious and unobtrusive, and certainly not aggressive. He earned the goodwill of all with whom he came in contact, and did everybody a good turn if it came his way. He was not accused of contemplating any harm to anybody, even to those that might have given him offence. I have had the pleasure of his friendship almost from the beginning of his career through practically the whole period as a colleague, and for a short period as a student when he was lecturing in dynamics for the Science section of the College. We got on throughout on very friendly terms, even in respect of matters in which we differed vitally from each other. I believe most of his

colleagues in the College could conscientiously say about as much. His students had nothing but affection for him, and the principals both the late Dr. Cook and Mr. Tait, under whom he worked had the highest regard for him. During the closing period of his career, he had to play the role of the University don in mathematics. He discharged his duties of this responsible position with conscientious care as he did those of the Board of Studies and examination board in mathematics at the Madras University during a long period of years. He was one of those remarkable men who conducted himself through his active life as an official with serene goodness and passed away honoured and cherished with affection and esteem by all those who came in contact with him.

From Prof. R. Srinivasan, Trivandrum.

My impression about Prof. M. T. Naraniengar is of a silent but strong man with deep devotion in his heart and plenty of intuition playing about him. A saintly figure with no ostentation in him, a benign smile playing about him, that is what stands before me when I think of him. When he came to Trivandrum to preside over a Mathematical Conference he was treated as a State Guest; but he arrived almost *incognito* and walked from the bus stand to Padmavilasam Palace where he was to stay—that shows the simple man he was. At a meeting of Examiners for the B A. (Hons.) Examination he had given a very interesting result which I had not noticed before. When I looked at him enquiringly he simply smiled and remarked, "Yes, it is a new result, it has not been noticed so far."

In personal life he was deeply religious; God was to him not a mere abstraction but a living Personality loving and guiding the entire Universe, and Mathematics was but one of the infinitely varied expressions of that Cosmic Intelligence.

From A. A. Krishnaswami Ayyangar, Mysore

A tall, slim personality with distiction in the eyes arrested my attention a quarter of a century ago, at the residence of Prof. P. V. Seshu Aiyar in Madras, when I was there to consult the professor on a little mathematical contribution which, however, never saw the light of day. I had just then emerged from my educational cocoon and little thought that my life was destined to be shaped by this man from afar. Prof. M. T. Naraniengar was the most monosyllabic of all the mathematicians I ever met, Indeed, the last monosyllable I exchanged with him only a few days before his demise, when we saw each other at the Durbar on the occasion of the *Pattabhishekam* ceremony of the present Ruler of Mysore. With all the outward absence of warmth he proved a delightful mathematical friend and I can vouchsafe several happy hours I have spent when I used to visit him in his house in Malleswaram. He always spoke with the precision of a mathematical formula and was incapable of ambiguities. There was no mis-

understanding him at any time and on any acount. Everything about him was trim, elegant, and standardised, including gestures, gait, dress, and finance. He could never be moved to anger, the extreme form of which in his case was just a smile of contempt.

Saintly in character and conduct and rigidly conservative and orthodox in scientific and academic outlook, he presented a vivid contrast to the fussy assertiveness and expansiveness of the heterodox type of professor. He did his duties, official as well as non-official, with a detachment, a devotion, and a profound sense of responsibility truly characteristic of him. Always religious and devoted to God, he took to the sacred literature with greater zeal than ever after retirement. His mathematics then got automatically metamorphosed into metaphysics and the last decade of his life was spent purely in charities and the service of God. A true son of Mathematics and a truer son of God!

V. Ramaswami Aiyar, P. V. Seshu Iyer and M. T. Naraniengar formed the illustrious trio who kindled and fostered the idea of research in mathematics in young India. The Indian Mathematical Journal owes not a little of its present status and position to the pioneering editorial, secretarial and presidential labours of this Dattatreya Avatar, three apparently distinct personalities rolled into one mathematical envelope. Though the persons are no longer here in flesh and blood, their spirit, example and blessing are bound to remain with us as beacon lights to the generations of Indian—especially South-Indian—mathematicians who will continue to cherish the *Journal of the Indian Mathematical Society*, its younger branch *The Mathematics Student*, and other branches that may yet sprout from the parent tree planted by the founders with so much love, care, and faith.

From S. Narayanan, Madura

I knew, M. T. Naraniengar prior to the year 1907, only by his contributions to the mathematical columns of the then 'Educational Times' published in London. About the year 1907 or 1908, I met him in the company of the late Mr. V. Ramaswami Iyer, the founder of our society—when all the members of the Society met in the Presidency College to welcome Mr. E H. Neville of Cambridge—when the late Mr. S. Ramanujam read his maiden-paper on 'Elliptic Functions and Modular transformations' which elecited the admiration of Mr. Neville. Since then, I was corresponding with him on mathematical problems old and new and he was always prompt in giving me the information I required. I remember V. R. telling me that M. T. N. and himself were working together for a few years while the former was at Bangalore and the latter at Mysore. It was then that celebrated Feuerbach's Theorem caught the fancy of the two young enthusiasts and immediately both vowed to find out an original proof of the theorem by pure geometry without the aid of projection, inversion or reciprocation. Both succeeded in giving proofs, and both bear the clear stamp of their originality.

From Dr. Ram Behari, Delhi.

Secretary of the Indian Mathematical Society.

I first made the acquaintance of Prof. M. T. Naraniengar in March 1921 at Lahore on the occasion of the Third Conference of the Indian Mathematical Society and met him again at Bombay in December 1932 on the occasion of the Silver Jubilee Conference. Being then a junior worker in the field of mathematical research, I thought that it would have been impertinent on my part to take the liberty of talking freely to such a senior mathematician, the distinguished foundation-editor of the Journal of the Society, and consequently I did not have the privilege of getting to know him very intimately. Nevertheless, I take the opportunity so kindly given to me by the Editor of *The Mathematics Student* to associate myself with others in paying my humble tribute to the memory of one who, since the very inception of the Society, for nearly twenty years, conducted almost single-handed the Journal of the Society which is the embodiment of the creative ability of our countrymen in the field of advanced mathematics.

The earlier volumes of the Journal show clearly that besides doing the editorial work, he by his own original and varied contributions kept the Journal going in its early stages when the idea of mathematical research was not very familiar and the task of running a mathematical journal was full of formidable difficulties. He was one of the first Secretaries, a member of the Managing Committee for many years and lastly President of the Society for two years. His devotion to the advancement of Mathematics, his high sense of duty and the saintliness and simplicity of his life have left a lasting impression on the minds of all those who were fortunate to have come into contact with him.

Prof. Naraniengar's death is a great loss to the Society. The best way in which his memory can be perpetuated by his friends and admirers is to strive to improve the Journal of the Indian Mathematical Society with which his name will always remain associated, so that it will continue to be a source of inspiration and usefulness for all Lovers of Mathematics and will keep the torch of mathematical knowledge ever burning.

From Dr. A. Narasinga Rao, Annamalainagar.

The Indian Mathematical Society was lucky in both its parents. Its Founder V. Ramaswami Aiyar, magnetic and forceful, with his boundless enthusiasm and his flair for reconstructing the most complicated mathematical structure from simple beginnings and by easy stages, was a Father of whom any learned Society might well be proud; while Prof. M. T. Naraniengar who nourished the Society into adolescence had all the virtues of the Indian Mother —intense devotion and an unlimited capacity for silent, unobtrusive, tireless work. Nor was its birth time badly chosen! For, barely 8 years earlier, a young graduate of Maharashtra who was later to become a President of the

Society had won the Blue Riband of British Mathematics and was being carried shoulder high and feted by the Indian Majlis at cambridge presided over by another future President of the Indian Mathematical Society †; while at home, the meteoric Ramanujan was even then pegging away at those problems of his that were soon to take the world by storm.

The end of the 19th century found Britain just emerging from her mathematical isolation and still steeped in the Tripos Tradition while India, true to her subject status, followed in the wake of Britain at a distance of (—10) years. With no traditions to fall back upon and little guidance from the Universities which were examining machines, the pioneers of the Society fell back on Synthetic and Analytic Geometry, the common factor of all mathematical education. In this field trodden hard by generations of treasure seekers from the hoary days of the Wise Man of Miletus, and with the use of tools as primitive as those of Euclid, it is gratifying to find the genius of awakened India bringing out gems of the purest ray serene. An instinctive feeling for the shape of truth guided them aright and the geometrical problems discussed in the early issues of the *Journal* are full of interest and beauty.

I have in my posession a number of letters addressed to V. Ramaswami Aiyar which show that he was in continuous correspondence with Mr. R. F. Davis and to a less extent with Prof. G. H. Hardy. These letters throw light on the genesis of the proofs of Feuerbach's Theorem by M. T. N. and V. R. to which repeated reference has been made. It would appear that there were 3 proofs of Feurbach's theorem which V. R. desired should appear together. Mr. Davis, however, suppressed his own as he thought "two proofs would be as much as Greenstreet would stand" and drafted the other two with a common figure "in a way acceptable to yourself and your friend N", and sent them for publication in the *Gazette*. He continues:—"I think they are both very good—I place yours decidedly first and I think N's runs it fairly close. I take credit to myself of drawing your attention to an old proof of my own (Nov. 1898) which has been the incentive of your discovering a far better proof on the same lines." A week later he is annoyed to find that M. T. N. has already offered a proof to The Educational Times, but the next letter dated 9th August 1908 suggests that the proof must have been a different version for he writes that it is complicated and hard to follow and continues "I fancy, however it will be considered that N's first attempt in the E. T. is the oldest and that it has inspired the later versions in the M. G. This is of course the wrong view and if I understand rightly your letters it was the consideration of your proof that led N to his own." It would thus appear that M. T. N and V. R. are as inseparable in their proof of Feuerbach's Theorem as in their activities connected with the Society.

† Dr. (then Mr.) R. P. Paranjpye (St. John's) was declared Senior wrangler in the Tripos of 1899 bracketed with Mr. George Birtwistle (Pembroke). Among those present at the complimentary dinner held on 20-6-1899 were Mr. R. Balakram (President) Mr. R. Narayanan, I.C.S., (Secretary), Mr. H. G. Gharpurey, I.C.S., and Messrs. J. Baptista, M. J. Mody, and B. C. Pal. In 1900 Mr. Balakram was 4th wrangler and next year H. G. Gharpurey was 7th. The wranglership was abolished later,

It cannot be claimed on behalf of Naraniengar that he initiated important developments in Mathematics, though his papers reveal a wide range of study and a firm handling of each problem. It is given to few to initiate such developments and, in any case, he was born too early in India for such a task. But he rendered an even more signal service to mathematics by nursing into adolescence in spite of formidable infantile ailments the *Journal* which was to canalize India's creative activity in matters mathematical and to give young Indians a training in the difficult art of formal presentation. This important task he carried on for nearly two decades—no mean part of a man's active life—wholeheartedly and without stint or reserve. Let us give him our tribute of admiration and respect wholeheartedly and without stint or reserve.

ANNOUNCEMENTS AND NEWS

The following gentlemen have been elected members of the Indian Mathematical Society:

1. K. S. Balakrishna Sastry Esq., M.A., Prof. of Mathematics, D. G N. College, Hyderabad (Sind);

2. Sita Ram Gupta Esq., M.A., Professor of Mathematics, Government College, Lahore.

Messrs. S. Mahadevan and Ram Behari have been elected Secretaries of the Indian Mathematical Society for the period ending August 1942.

The Indian Mathematical Society has accepted the invitation of the Aligarh University to hold its next conference at Aligarh. The dates fixed for the session are 27th, 28th and 29th of December 1941. (Aligarh University observes Fridays and not Sundays as holidays). Papers intended for the conference, together with two brief abstracts of each should be sent to Prof. A. Narasinga Rao, Annamalainagar P.O , South India, so as to be received before the 1st of October, 1941.

We offer our hearty congratulations to Dr. Homi J. Bhabha, Reader in Theoretical Physics at the Indian Institute of Science, Bangalore, who has been elected a Fellow of the Royal Society in recognition of his outstanding "contributions to the understanding of the cosmic ray phenomena and the fundamental theory of the elementary atomic particle." Dr. Bhabha who is now 31 years old, took the Cambridge Mathematical Tripos in 1930 and has travelled widely over Europe and worked with many of the outstanding Physicists of today.

We felicitate Sir C. V. Raman, on being the recipient of the Medal awarded by the Franklin Institute of Pennsylvania "in recognition of his many brilliant contributions to Physical science and of his leadership in the renaissance of scientific work and scientific education that has occurred in India during the last 30 years". The list of previous recipients of this medal includes Edison, Einstein, Michaelson, Planck, Rutherford and Zeeman. It is a great honour coming as it does from a country in the forefront of Mathematical and Physical research. Though the cardinal number of academic distinctions showered on Sir Raman must be transfinite and may not be affected by adding one more, it affects the ordinal number as being the last. We wish Sir Raman a triumphant progress from $\omega+1$ to $\omega \cdot 2$

Sir Ziauddin Ahmad has been appointed Vice Chancellor of the Muslim University, Aligarh.

Dr. B. Ramamurti of Ajmere delivered a popular lecture on "Mathematics the Queen of the Sciences" at the All India Educational conference at Udaipur,

Prof. P. C. Mahalanobis, Director of the Statistical Institute, Calcutta, has been elected President of the Mathematics and Statistics Section of the next session of the Indian Science Congress which is due to meet at Dacca between the 2nd and 8th January 1942.

The Twenty-eighth Session of the Indian Science Congress

Benares

The Indian Science Congress met at Benares between the 2nd and the 8th January 1941. The session was opened by His Excellency the Governor of the United Provinces. He pointed out how the exigencies of the war provided a good opportunity for industrial expansion in India, and hoped that when victory was won, science would be harnessed to progress and prosperity. Sir S. Radhakrishnan, in welcoming the delegates, stressed the need for scientific advance and industrial development, but pleaded for scientific achievement without a sacrifice of spiritual values. Sir Aredeshier Lal, the General President pointed out the value of research in Industry, welcomed the appointment of the Board of Scientific research, and echoed the words of Lord Rutherford for a greater amount of freedom for scientists in such bodies.

The next day onwards, the conference split into 14 sections, which met every morning; Discussion and symposia were held in the afternoon, and were followed by Tea-parties, popular lectures and entertainments. Dr. K. S. Krishnan, F. R. S. delivered a fascinating lecture on 'The Earth as a giant magnet'.

In the section devoted to Mathematics and Statistics, 36 papers were read on a variety of topics. Besides these, three symposia were held. The first one related to Boundary value problems in Differential Equations. Rev. Fr. Racine (Madras) dealt with some recent generalisations of the Dirichlet problem, and Dr. Siddiqi (Hyderabad) with modern methods for the solution of boundary value problems. Profs. S. C. Dhar (Nagpur) and N. M. Basu (Dacca) also participated in the discussion. Diophantine Analysis was the topic of the second symposium. Dr. T. Vijayaraghavan (Dacca) gave a lucid historical survey of the subject. Dr. Pillai (Annamalainagar) confined himself to Exponential Diophantine Equations, while Dr. Chowla (Lahore) spoke on Recent progress in the Additive Theory of Numbers. Prof N. M. Basu and Mr. Shastri (Benares) also spoke on their investigations in the Theory of Numbers. Dr. R. C. Majumdar opened the symposium on the theory of stellar structure and Prof K. S. Krishnan on the "Structure of Solids." These were joint meetings with the Physics section. The average attendance at the Mathematics Sectional meetings varied between 50 and 60.

The President of the Mathematics Section Dr. Siddiqi delivered a learned and instructive address on "Functional Analysis and Mathematical Physics." He pointed out how progress in certain branches of Modern Mathematics and progress in Modern Physics have been influencing each other to their mutual

advantage. The theory of differential equations total and partial, then the theory of integral equations linear and non-linear, the theory of their eigen-functions and eigen-values, the representation of functions in Hilbert Space, the Calculus of variations and the concept of functionals and functional operators, these were links in a chain of progress achieving more and more generality, suggested by investigations in modern physics, particularly in the region of atomic physics.

B. RAMAMURTI.
S. C. DHAR.

The Fourth Session of the Indian Statistical Conference

Benares.

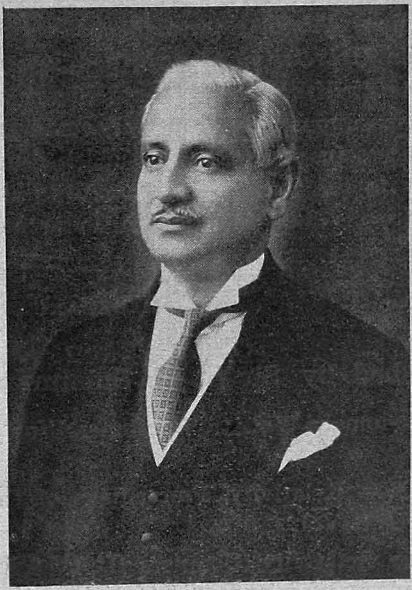
The Statistical conference was held along with the Science Congress at Benares. Sir Bajpai, the President of the Session pointed out that the collection and interpretation of Statistics will be attended with greater usefulness if their purposes are explained and processes made more intelligible to the ordinary citizen. Sir Radhakrishnan appreciated the usefulness of statistics, provided its methods are rightly and scientifically applied. Prof. P. C. Mahalonobis, the General Secretary, gave a comprehensive account of the work of the Statistical Institute at Calcutta, particularly in the theory and technique of sample surveys, and of the various statistical inquiries in which it is engaged. He also pointed out the need for sample surveys before and after any scheme, in order to measure its success. About 50 papers were submitted and read, comprising many branches of statistics, theoretical and applied, in various fields like Anthropometry, Agriculture, Economics, Railways and Astrology. Joint sessions were held with the Science Congress in Anthropology, Agriculture, Medicine and Mathematics. Theoretical Statistics was taken up in the joint session with Mathematics. S. N. Roy (Calcutta) of the Statistical Institute gave a clear account of the recent progress in the problem of discrimination and classification of samples. D. D. Kosambi (Poona) read his paper on the distribution of the Z-statistics in samples from non-normal populations and B. Ramamurti (Ajmer) read his paper on Maximal confounded sets in Agricultural experiments. There was also a long discussion in the teaching of statistics in which most of the statisticians and teachers of statistics participated and opinion was almost unanimous on the need for providing the requisite and appropriate in mathematical background, in any advanced course in statistics.

B. RAMAMURTI.

OBITUARY

SIR SHAH MOHAMMAD SULAIMAN

Sir Shah Mohammad Sulaiman was born at Jaunpur in the United Provinces of Agra and Oudh on February 3, 1886. He had his school education at Jaunpur, and, after passing the Matriculation Examination from there, joined the Muir Central College, Allahabad. His educational career was exceptionally brilliant. He stood first in the Allahabad University in the B.A. examination and was awarded a Government scholarship in 1906 for higher study in England. He then joined the Christ Church College, Cambridge, took the Mathematical Tripos in 1909, Law Tripos in 1910, was called to the Bar, Middle Temple, in 1910, and took the degree of Doctor of Laws from the Dublin University in 1911.



Sir Shah Sulaiman started legal practice at the Allahabad High Court in 1912 and within a short period of eight years, at the age of 34, he was called to the Bench and became the first Indian Chief Justice of the Allahabad High Court. He was knighted in 1929. Later, when the Federal Court of India was constituted at Delhi, he was appointed one of its three judges. He was regarded as an authority on Constitutional Law.

Sir Shah Sulaiman was a great educationist. He was Vice-Chancellor of the Muslim University, Aligarh, twice, once in 1929 and then again for several years till the time of his death which occurred on 12th March, 1941.

In spite of his legal and other administrative duties he continued to maintain his interest in Mathematics in his rare moments of leisure, and contributed several papers in support of a new Theory of Relativity which formed the subject of discussion in various scientific journals.

Besides delivering several scientific addresses, he published about 31 scientific papers in Indian and foreign journals. His 'New Theory of Relativity' has attracted the notice of scientists all over the world. The Allahabad University conferred the degree of D.Sc. on him and the National Institute elected him a Fellow. He was the President of the National Academy of Sciences, India, and was an active member of the Managing Committee of the Indian Mathematical Society.

As a man, Sir Shah Sulaiman was an embodiment of culture, refinement, and gentleness. His boundless energy and unflagging zeal for work made a lasting impression on everybody who came in contact with him. He combined the exuberance of youth with the maturity of age. He was modest and courteous and was admired and loved by all fellow-workers in the field of science or law.

In Sir Shah Sulaiman's death India has lost an exceptionally gifted lawyer, a keen scientist and a great patron of learning. About a fortnight before his death he, as Vice-Chancellor of the Muslim University, Aligarh, had invited the Indian Mathematical Society to hold its Twelfth Conference at Aligarh on December 27-29, 1941. He had promised his support in various ways, and the Society was looking forward to a very successful session. His sudden death before the Aligarh Session is a great blow to the Society. His death will be deeply mourned by a large number of friends and admirers in India and abroad.

Delhi. } RAM BEHARI.

A List of Scientific Papers Published by Sir S. M. Sulaiman.

1. A Unified Theory of Physical Phenomena Vol. 1. The Inner World. 23. 9. 1933.
2. The Mathematical Theory of a New Relativity, Proceedings of the National Academy of Sciences, U. P. Vol. 4. 1934.
3. The Mathematical Theory of a New Relativity (Chapters III, IV & V) Proceedings of the National Academy of Sciences, U. P. Vol. 4. 1935.
4. The Mathematical Theory of a New Relativity (Chapters VI to IX) Proceedings of the National Academy of Sciences, U. P. Vol. 5. 1937,
5. Reply to Mr. D. R. Sharma's criticisms, (Not available).
6. Reply to S. C. Damle's note on the New Theory of Relativity, Proceedings of the National Academy of Sciences, Vol. 6. 1936.
7. The Mathematical Theory of a New Relativity, Current Science, Vol. 4. 1935.
8. Has the Theory of Relativity been verified? Science and Culture, Vol. 1. 1936.
9. The Expanding Universe. Indian Physico-Mathematical Journal. Vol. 7. 1936.
10. The Mathematical Theory of a New Relativity Chapter XIII, Proceedings of the National Academy of Sciences, U. P. Vol 6. 1936.
11. The Mathematical Theory of a New Relativity, Chapter XIV. Proceedings of the National Academy of Sciences, U. P. Vol. 6. 1936.
12. The Law of Gravitation, Indian Physico-Mathematical Journal. Vol. 7. 1936.
13. Relativity on Trial. Science and Culture, Vol. 2. 1937.

14. The Mathematical Theory of a New Relativity, Chapter XV Proceedings of the National Academy of Sciences, Vol. 7. 1937.
15. Rotating Radions. Proceedings of the National Academy of Sciences, Vol. 8. 1937.
16. Relativity Tested. Science and Culture. Vol. 3, 1937.
17. Gravitation. 'Southern Stars' (New Zealand) Vol. 3, 1937.
18. The Rotational Theory of Light. Indian Physico-Mathematical Journal. Vol. 8, 1937.
19. The Mathematical Theory of a New Relativity—Reply to Zahur Hussain's criticisms. Current Science, Vol. 6, 1938.
20. Sulaiman's Predictions. Science and Culture. Vol. 6, 1938.
21. Levi-Civita's Formulae for Two Bodies. National Institute. 1938.
22. The Theory of a New Relativity Chapter XVI. Proceedings of the National Academy of Sciences, Vol. 9, 1939.
23. The Solution of Einstein's Orbital Equation. Phil. Mag. Vol. 26, 1938.
24. The Astronomical Consequences of Relativistic Two-Body Problem. Phil. Mag. Vol. 28, 1939.
25. The Cambridge Observatories and Total Solar Eclipses. Science and Culture. Vol. 5, 1939.
26 The Royal College of Science and the Next Solar Eclipse. Science and Culture. Vol. 5, 1940.
27. Reply to Mr. Roy's paper. Proceedings of the National Academy of Sciences, U. P. 1940.
28. Reply to Dr. Japolsky. Proceedings of the National Academy of Sciences, U. P. Vol. 10, 1940.
29. Matter and Light. (Indian Review) 1940.
30. Modification of a Relativity Postulate. Phil. Mag. Vol. 30, 1940.
31. The Nature of light and Matter 'Southern Stars' (New Zealand) Vol. 6, December 1940.

Scientific Addresses.

1. Presidential Address delivered at the Academy of Sciences at Allahabad on the 20th January, 1934.
2. Reality of Matter and Mind, Allahabad, 15th March, 1937.
3. Presidential Address, Idara-i-Maarif-i-Islamia, Delhi, 26th December, 1938.
4. The Ultimate Structure of Matter—National Academy of Sciences, 4th February, 1939.
5. Address at the National Academy of Sciences, Allahabad, 13th January, 1940.
6. Presidential Address, National Academy of Sciences, Delhi Session, 1941.

REVIEWS

C. DE LA VALLÉE POUSSIN—*Les nouvelles méthodes de la théorie du Potentiel et le Problème généralisé de Dirichlet.*—Hermann et Cie, Paris 1937 (Actualites scientifiques et industrielles No. 578).

Prof. de La Vallée Poussin has condensed in the 46 pages of this fascicle a very rich harvest of results obtained by him and some other mathematicians during the past ten years. I have already reviewed in *The Mathematics Student* (Cf. March 1940, p. 44) two other fascicles of the same series dealing with the same subject. Although this fascicle can be read independently, it really forms with N°571 and N°660 *one* treatise of the modern theory of Potentials. Those who have no access to these fascicles may refer to *The Mathematical Gazette* (Vol. 22 1938, p. 17-36), where the same theory is treated but more summarily, by Prof. de La Vallée Poussin.

I shall restrict myself to making a brief analysis of the eight paragraphs of the booklet under review.

§I is an introduction. The generalized Dirichlet problem has been set by N. Wiener in 1924. Instead of a domain whose boundary satisfies the classical conditions of regularity (°), he considers any open and connected domain. A continuous function is given on its boundary. The problem is to determine a function harmonic in the domain and whose boundary values are equal to the given function. In general the problem is impossible but it becomes possible if a certain set of points, the "irregular points" are excluded.

N. Wiener succeeded in constructing a function satisfying the above conditions. After him, Kellog, Evans and Vasilesco developed a theory of sets of nul capacity which made it possible to characterize the irregular points and to state a theorem of uniqueness. At the same time Prof. de La Vallée Poussin extending the results of Poincare proposed a new method of "balayage" leading to the most elegant solution of the Dirichlet problem in the most general case. These different theories form now a coherent theory of Potentials.

§II is devoted to the theory of the Stieltjes-Lebesgue integral. By this integral potentials can be defined without using the notion of density. This paragraph can hardly be understood when one has not read the celebrated book "*Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire*" by the same author (Borel series).

(°) Cf. N. M. Gunther,: *La theorie du Potentiel et ses applications aux problemes fondamentaux de la physique mathematique*—Paris, Gauthier Villars, 1934 (Borel Series).

§III sets forth the notion of Newtonian Potential defined by the Stieltjes-Lebesgue integral.

In §IV the notion of capacity is introduced and briefly explained.

§V and §VI contain the main results of the theory, almost all of them due to the new method of balayage. The balayage (sweeping out method) is an operation by which to masses da distributed over a set A are substituted masses $d\mu$ distributed over a closed set E—in general the boundary of A—when E contains no mass da. The potential must remain unchanged on E except at points forming a set of nul capacity. The existence and the uniqueness of such a distribution over E is proved by using Gauss' integral of energy but expressed as a Stietjes Lebesgue integral. The distribution of the balayage minimizes a certain Gauss' integral. This is one of the most important theorems of the new theory.

Now let E be an open set, bounded or unbounded, whose boundary F is bounded, and let U (Q) be a continuous function defined on F. Let P be a point interior to E and Q a point of F. Let $d\mu(Q, P)$ be the distribution defined on F by the balayage of a unit mass at P. Then

$$U(P) = \int_E U(Q)\, d\mu(Q, P)$$

gives the complete solution of the generalized Dirichlet problem for E relatively to the boundary values U(Q).

VIII gives an interesting generalization of the previous problem by considering the case of a boundary F whose accessible points are accessible with a certain order of multiplicity.

The Dirichlet problem has thus been treated with such a degree of generality and such an elegance that one cannot help comparing this new theory for its "fini" to the Elements of Euclid. Will it be possible to perfect it still further? At any rate after the research work of Prof. de La Vallée Poussin, this is now a very difficult task.

M. R. SIDDIQI, Osmania University, Hyderabad.—*Boundary Problems in Non-Linear Partial Differential Equations.*—Lucknow University Studies, No. XI 1939.

Dr. Siddiqi delivered at the Lucknow University in 1938 a course of lectures which is now published in the very valuable series of Lucknow University Studies.

In his introduction, Dr. Siddiqi describes what he calls the "two different methods of attack" of the boundary problems with non linear equations. I should have mentioned three methods, for the admirable work of J. Schauder and J. Leray has resulted in a new, very original, method of solution of the problem in the case of

elliptic and hyperbolic equations. This method is now entirely distinct from Green's method, and the "Fixpunktsatz" generalized by J. Schauder to Banach spaces is a great improvement on the method of successive approximations. Anyhow Dr. Siddiqi has devoted his monograph to a short account of two methods only: the so-called "Fourier method" and the method of the Green's function. The second one is briefly described in Chapters III and IV. In Chapter III the method is defined and applied to non-linear differential equations of the elliptic type; in Chapter IV it is applied to non-linear equations of the parabolic type. The other chapters namely Chapters I, II, and V deal with the Fourier method. In these chapters Dr. Siddiqi has written a very lucid account of his own research work. The idea was suggested to him by Lichtenstein to study some non-linear parabolic and hyperbolic equations by writing the solution in the form of a Fourier series, or of a series of eigen-functions. The equations considered in the volume under review are the following:

$$\frac{\partial}{\partial u}\left\{p(x)\frac{\partial u}{\partial x}\right\} - \frac{\partial u}{\partial t} = P\left(x, t, u, \frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = u^2$$

$$\frac{\partial}{\partial x}\left\{p(x)\frac{\partial u}{\partial x}\right\} - \frac{\partial^2 u}{\partial t^2} = \sum_{I}^{\infty} P_r(x, t)\, u^r = P(x, t, u)$$

$$\frac{\partial}{\partial x}\left\{p(x)\frac{\partial u}{\partial x}\right\} - \frac{\partial^2 u}{\partial t^2} = P\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right)$$

In the first one $p(x)$ is a function which is greater than 1 in the interval $(0, \pi)$, continuous and uniformly bounded along with its first derivative in the same interval. P is supposed to be analytic in u and $\frac{\partial u}{\partial x}$ continuous, as also $\frac{\partial P}{\partial x}$ and $\frac{\partial^2 P}{\partial x^2}$, these quantities being uniformly bounded in the same interval for $t \geqslant 0$. In the third equation P satisfies the same conditions with regard to x, t, and u. In the fourth equation P is continuous with regard to all the variables and satisfies with regard to the last three a Lipschitz condition. Further

$$P(0, t, 0, \alpha, 0) = P(\pi, t, 0, \alpha, 0) = 0.$$

Some classical types of boundary values are considered. In each case the method consists essentially in assuming that under certain conditions, the solution can be represented by a series of Sturm-Liouville eigen-functions. The coefficients of those series are determined

as solutions of an infinite system of non linear integral or integro-differential equations. These systems have been carefully studied by Dr. Siddiqi and Mr. S. Minakshisundarm whose work is referred to in Chapter V. They yield the solution of some boundary problems.

In such a short book it was obviously impossible to give more than a very brief account of the Green's and of the Fourier methods. Yet I think that the Green's method could have been more elaborately explained. The conditions which the boundary of the domain must satisfy are, for instance, not at all treated. The reader is simply given a reference to the German Encyclopedia. But these conditions are so important and the results of Liapounoff in this field so elegant that they could have been profitably included. With the recent work of Kellog, Evans, Vasileco and de La Vallèe Poussin, boundary value problems have been solved with more general conditions in the case of elliptic equations. But the work of Liapounoff will remain for a long time important and classical.

In the chapters treating the Fourier method, particularly in the first one, a very long account is given of certain infinite system of non-linear integral equations. Of course Dr. Siddiqi was addressing students for whom such a detailed exposition was necessary. I think that a short monograph could have restricted itself to a shorter development of calculations which do not offer any theoretical difficulty. Perhaps it would have been possible to insist more on the conditions of the representation of a function by a series of eigenfunctions. The results summarized from page 21 to page 26 presuppose delicate discussions about the nature of such a representation. Of course we are referred to an article of Lichtenstein; but I am sure that some analysis of this article would not have been out of place.

The book ends with a bibliography which, Dr. Siddiqui points out, is not exhaustive. It aims at giving only the most representative articles or volumes on the subject. I must confess to my surprise not to have found in it a reference to the admirable monograph due to G. Ascoli, P. Burgatti and G. Giraud, published in 1935 in the series of "Publicazioni della R. Scuola Normal Superiore di Pisa" under the title "Equazioni alle derivate partialai dei tipi ellittico e parabolico." Although most of it is devoted to the linear equations, much is said which is of great use in the case of non-linear equations and in the last chapter of the third part, written by G. Giraud, the non-linear equations of the elliptic and parabolic types are treated.

These questions in the case of non-linear parabolic equations have been treated lately by the method of Schauder by Mr. S. Minakshisundaram, in two still unpublished memoirs, which extend the results of Chapter I of this book to a much more general case. I hope that the very clear and interesting resumé which Dr. Siddiqi has given of his own research work will stimulate many other valuable researches like those of Mr. S. Minakshisundaram.

C. RACINE.

SITA RAM GUPTA, *Elementary Analytical Dynamics of a Particle*, Atma Ram & Sons, Lahore. 1939, pp. 354, Price Rs. 3-8-0.

The author who is a lecturer in Applied Mathematics in the Government College, Lahore, has brought out this third edition of his book, first published in 1931. As its title indicates, the range of topics discussed include Newton's Laws of Motion, Motion in a straight line (constant acceleration, simple harmonic motion) and on a circle (uniform motion, motion of a heavy particle in contact with a smooth vertical circle), projectiles, impact etc. Among more advanced topics one finds elliptic harmonic motion, cycloidal pendulum and planetary motion. The last chapter deals with moments of inertia and the rotation of a body about a fixed axis. The book should thus meet the requirements of students of the B. A. or B. Sc. Pass classes of Indian Universities, or of a first course in Dynamics for those in Honours classes.

While books dealing with this range of topics do not presuppose a knowledge of the calculus, and give a treatment by elementry methods, the author makes use of the calculus from the very start. This is all to the good, since the students in the Pass classes will be doing calculus as part of their course, and in many universities will have already had a first course in calculus in their intermediate classes. Besides, such notions as velocity or accleration at a point are really notions belonging to the infinitesimal calculus and most amenable to discussion by its methods.

The treatment is on the whole good, but the handling of fundamentals is not above criticism. The definition of the instantaneous velocity or acceleration is best approached through the notion of average speed or acceleration taken over a small interval. The author is guilty of defining in a circle when he writes that the "actual speed at a given point at a given instant is measured by the distance which would be described in a unit of time by a body having the

same speed as that which the body has at that instant." Again the use of the worlds "however small" or "however short" is unnecessary in dealing with uniform motion, and there would be nothing lost in their omission in statement like "in equal intervals of time, however short, the displacements of the particle are equal and in the same direction" (page 3). In fact it is the other way; for, deviations from uniformity are best detected by comparing the distances moved by the particle in two equal but *long* intervals of time.

There are quite a number of worked examples and of exercises for the student. The book should prove useful as a text book in Dynamics for the B. A. or B. Sc. Pass classes and deserves every encouragement.

A. N. RAO.

MUKAND LAL DHAWAN, Retd. Professor of Mathematics, Govt. College, Lahore: *Contributions to the Fundamentals of Arithmetic* (Published by R. S. Mukand Lal Dhawan, Lahore. Price 0-7-0).

In this booklet of 70 pages, the author claims to have given shorter and more rapid methods for carrying out the processes of multiplication, division, decimalisation and extraction of square and higher roots. It is very doubtful whether the new methods explained in the book have any decided advantage over the classical methods. For instance, taking multiplication, the new method will be effective only if the pupil can add rapidly in succession numbers lying between 1 and 81 while the classical method requires only the addition, in succession, of numbers between 1 and 9. In some special cases the new methods might give the result more quickly than the older ones, but this does not appear to be the case universally, especially when the numbers involved are very large. Moreover, the methods are not suited for elementary teaching since only a pupil above the matriculation standard can grasp the processes explained in the book. Nor can it be claimed that the booklet is useful for persons whose business in life is to deal with large calculations, since mechanical means (machines and multiplication tables) have been invented to facilitate such calculations.

The systematisation of the tests of divisibility by the use of "associate numbers" is the only noteworthy point of interest in the book. The title, 'Fundamentals of Arithmetic' is rather misleading from the point of view of higher mathematics, since it might be taken to mean a study of the foundations of abstract integers or irrational numbers. "Shorter methods in elementary arithmetic" would have been a happier title.

V. G.

UNSOLVED QUESTIONS FOR WHICH SOLUTIONS ARE INVITED

Journal of the Indian Mathematical Society

478. (M. T. NARANIENGAR). If S, H be the foci and O the centre of the maximum inscribed ellipse of a triangle ABC, prove that

$$AS \cdot BS \cdot CS + AH \cdot BH \cdot CH = 2AO \cdot BO \cdot CO$$

614. (M. T. NARANIENGAR). If $u_n = \left(\frac{1 \cdot 3 \cdot 5 \cdots\cdots 2n-1}{2 \cdot 4 \cdot 6 \cdots\cdots 2n}\right)^2$, prove that

$$u_n = 4^{2n} u_{2n} - (4n-2) 4^{2n-1} u_{2n-1} + \frac{(4n-3)(4n-4)}{1 \cdot 2} 4^{2n-2} u_{2n-2} - \cdots\cdots$$

The Educational Times

16617. (M. T. NARANIENGAR). Prove that the vertices of the harmonic triangle of the Feurbach quadrilateral lie on the sides of the pedal triangle.

16629. (M. T. NARANIENGAR)—Suggested by Question 16392.

In a limacon, PQR is an inscribed triangle having the focus for the orthocentre. Prove that the circumcircle of PQR touches two fixed circles, having their centres on the axis of the limacon, and that the locus of the centre of PQR is a conic with the centres of the fixed circles for foci.

17232. (M. T. NARANIENGAR). If the normal at P meets the parabola in Q, and the normal at Q the axis in G, show that APQG is cyclic.